Objective of the course

Introduce how to use linear algebra techniques to solve practical problems in:

Image processing, Differential equations, Difference equations, Quantum Computing, Optimization, Deep Learning, etc.

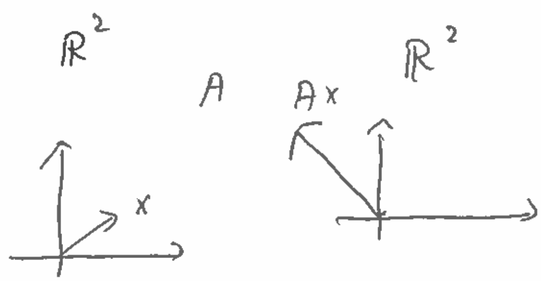
A simple description of Deep Learning and Neural Network model.

$x \to \boxed{F} \to F(x)$

$\Uparrow$

$\{0, 1, \ldots, 9\}$

- A simple example. Identify the images of $\boxed{0, \ldots, 9}$ using training data $x_1, \ldots, x_n$.
- Apply functions $F(x)$ so that it will correctly identify the outcome.
- It turns out that functions of the form $F(x) = L(R(L(R(\cdots(x)))))$, where $\boxed{L(x) = Ax + b}$ and $R(x) = (\max(0, x_1), \ldots, \max(0, x_n))^T$ for $x = (x_1, \ldots, x_n)^T$ work well.
- In the neural network setting, one uses the input v to adjust $L_k = A_k v_{k-1} + b_k$, to produce a new hidden layer.
- The composite function $F(v) = L_k(R(L_{k-1}(R \cdots (v))))$ adds depth to the network and leads to more successful model.

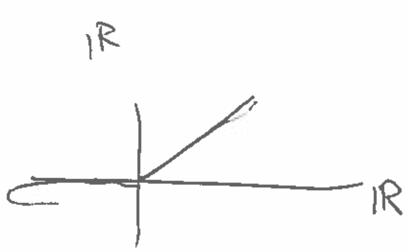


$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \to \begin{pmatrix} \max(0, x_1) \\ \vdots \\ \end{pmatrix}$$

$x \mapsto Ax$

$A(x+y) = Ax + Ay$

$A(\lambda x) = \lambda Ax$



1 1 1 1 $\to$ 1

9 9 $\to$ 9

8 8 8 $\to$ 8

Basic notation and background $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_5$

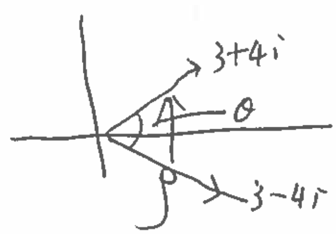
- $M_n(\mathbb{F})$, $M_{m,n}(\mathbb{F})$ are the set of $n \times n$ and $m \times n$ matrices over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ (or a general field).
- Basic operations of complex numbers are assumed.
  $z = x + iy = \rho e^{i\theta}$, $\rho = |z| = |\bar{z}z|^{1/2} = \sqrt{x^2+y^2}$, $\bar{z} = x - iy$, $z_1 + z_2, z_1 z_2, z_1/z_2$ if $z_2 \neq 0$.
- $\mathbb{F}^n$ is the set of column vectors of length n with entries in $\mathbb{F}$.
- If $\mathbb{F}$ is clear, we use the notation $M_n, M_{m,n}$. $\longrightarrow$
- Let $A \in M_{m,n}$. Then $A^T \in M_{n,m}$. For complex matrix A, we have $\bar{A}$ and $A^* = (\bar{A})^T = \overline{A^T}$.

Example.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 3+i \end{bmatrix} \in M_2(\mathbb{C})$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \in M_{2,3}(\mathbb{R})$$

Example $z = 3+i4 = 5(\cos\theta + i\sin\theta)$



$\cos\theta = \frac{3}{5}$, $\sin\theta = \frac{4}{5}$

$5 = \sqrt{3^2+4^2}$

$|z| = |\bar{z}z|^{1/2} = \sqrt{x^2+y^2}$

Example: $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 5 \\ 6 \end{bmatrix} \in \mathbb{R}^5$

Example $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $x^2+1=0$ has complex eigenvalues. $i, -i$.

Example. $A = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \end{bmatrix}$ $A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 5 & 6 \end{bmatrix}$

Example $A = \begin{bmatrix} 1+i & 2 & i \\ 3 & 4 & 0 \end{bmatrix}$, $\bar{A} = \begin{bmatrix} 1-i & 2 & -i \\ 3 & 4 & 0 \end{bmatrix}$ $A^* = \begin{bmatrix} 1-i & 3 \\ 2 & 4 \\ -i & 0 \end{bmatrix}$

$$2x_1 + 3x_2 = b_1$$
$$2x_1 + 4x_2 = b_2$$
$$3x_1 + 7x_2 = b_3$$

I.1 A close look at $Ax = b$

$$\begin{bmatrix} 2 & 3 \\ 2 & 4 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

- Linear equations, elementary row operations, solution sets.

  Recall. Let $A \in M_{m,n}(\mathbb{F}), b \in \mathbb{F}^n$, $Ax = b$.

  Find $E_r \cdots E_1[A|b] = [\tilde{A}|\tilde{b}]$ in row echelon form, $E_r \cdots E_1[A|I_n] = [I_n|A^{-1}]$.

$$x_1 \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 4 \\ 7 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

- Matrices, column space, row space, null space, ranks. → rank(A) = # non-zero rows of

  Recall. Let $A \in M_{m,n}(\mathbb{F})$, and $E_r \cdots E_1 A = [\tilde{A}]$ in echelon form. The row echelon form of A $= \dim C(A^T) = \dim C(A)$

  Then we can find the bases for column space, row space, and null space, and the rank of A.

$A \in M_{m,n}$, null space $N(A) = \{x \in \mathbb{F}^n : Ax = 0\}$

Interpretation of $Ax = b$.

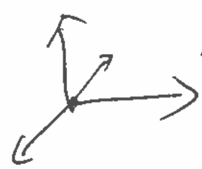
Example Let $A = [A_1 A_2] \in M_{3,2}, x = (x_1, x_2)^T, b \in \mathbb{F}^3$.

Then $Ax = b$ means $b = x_1 A_1 + x_2 A_2$.

All combination of A_1, A_2 form the column space.

The equation $Ax = b$ is solvable means that b is in the column space.

In general, if $A \in M_{3,n}$, the column space can be of dimensions $0, 1, 2, 3$.

$\mathbb{R}^3$ 

$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ $C(A) = \{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\}$

$A = \begin{bmatrix} 1 & 2 & 5 \\ 1 & 2 & 5 \\ 1 & 2 & 5 \end{bmatrix}$ $C(A) = 1$ is a line. $= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 5 \end{bmatrix}$

All these comments hold for the general case: $Ax = b$.

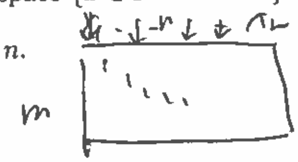
For example, if $A = [A_1|\cdots|A_n] \in M_{m,n}(\mathbb{F})$ and $Ax = b \in \mathbb{F}^m$ then $b = x_1 A_1 + \cdots + x_n A_n$.

If $Ax = 0 \in \mathbb{F}^m$ has non-trivial solution then $\{A_1, \ldots, A_n\}$ is linearly independent, i.e., there is $x_1, \ldots, x_n$ not all zero such that $x_1 A_1 + \cdots + A_n = 0$.

The null space of $A \in M_{m,n}$ is the set/subspace $\{x \in \mathbb{F}^n : Ax = 0\}$.

Theorem rank + null space dimension $= n$.

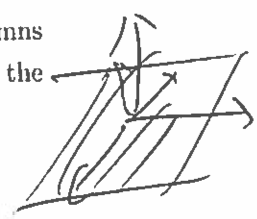
$A \in M_{m,n}$ pivoting one + rest $= n$



$A = \begin{bmatrix} 2 & 3 \\ 2 & 4 \\ 3 & 7 \end{bmatrix}$

$C(A)$

Proposition If $A \in M_{m,n}$ has rank r, then $A = CR$, where $C \in M_{m,r}$ with independent columns forming a basis for the column space, and $R \in M_{r,n}$ has independent rows forming a basis for the row space. So, the row rank and column rank of A are the same.



Remark The result is useful for low rank factorization.

There will be better factorization, namely, the singular value decomposition.