$$5\begin{bmatrix}1\\3\end{bmatrix}+6\begin{bmatrix}2\\4\end{bmatrix}=\begin{bmatrix}1&2\\3&4\end{bmatrix}\begin{bmatrix}5\\6\end{bmatrix}$$

$$\begin{bmatrix}1&2\\3&4\end{bmatrix}\begin{bmatrix}5&7\\6&8\end{bmatrix}=\left[\begin{array}{c|c} \quad & \quad \\ \quad & \quad \end{array}\right]$$

$$\begin{bmatrix}1&2\\3&4\end{bmatrix}\begin{bmatrix}7\\8\end{bmatrix}=7\begin{bmatrix}1\\3\end{bmatrix}+8\begin{bmatrix}2\\4\end{bmatrix}$$

$$\begin{bmatrix}1&2\\3&4\end{bmatrix}\begin{bmatrix}5&7\\6&8\end{bmatrix}=\left[\begin{array}{cc} \quad & \quad \\ \hline \quad & \quad \end{array}\right]$$

$$\begin{bmatrix}1&2\end{bmatrix}\begin{bmatrix}5&7\\6&8\end{bmatrix}=\begin{bmatrix}5&7\end{bmatrix}+2\begin{bmatrix}6&8\end{bmatrix}$$

$$\begin{bmatrix}3&4\end{bmatrix}\begin{bmatrix}5&7\\6&8\end{bmatrix}=3\begin{bmatrix}5,&7\end{bmatrix}+4\begin{bmatrix}6&8\end{bmatrix}$$

$$\begin{bmatrix}1&2\\3&4\end{bmatrix}\begin{bmatrix}5&7\\6&8\end{bmatrix}=\begin{bmatrix}1\\3\end{bmatrix}\begin{bmatrix}5&6\end{bmatrix}+\begin{bmatrix}2\\4\end{bmatrix}\begin{bmatrix}7&8\end{bmatrix}.$$

$$A = \begin{bmatrix} 1 & 3 & 8 \\ 1 & 2 & 6 \\ 0 & 1 & 2 \end{bmatrix} = \overset{C}{\begin{bmatrix} 1 & 3 \\ 1 & 2 \\ 0 & 1 \end{bmatrix}} \overset{R}{\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

φ

$\boxed{A \text{ is } m \times n}$

$C\begin{bmatrix}1\\0\end{bmatrix}$

$C\begin{bmatrix}0\\1\end{bmatrix}$

Column space

$Col(A)$

$= \text{span}\left\{ \begin{bmatrix}1\\1\\0\end{bmatrix} \begin{bmatrix}3\\2\\1\end{bmatrix} \right\}$ in $\mathbb{R}^m$

$C\begin{bmatrix}2\\2\end{bmatrix} = 2\begin{bmatrix}1\\1\\0\end{bmatrix} + 2\begin{bmatrix}3\\2\\1\end{bmatrix}$

$Row(A) = C(A^T) = \text{span}\left\{ \begin{bmatrix}1\\0\\2\end{bmatrix} \begin{bmatrix}0\\1\\2\end{bmatrix} \right\}$ in $\mathbb{R}^n$

$Null(A) = \{ x \in \mathbb{F}^n : \underset{m}{A} \overset{n}{x} = 0 \text{ in } \mathbb{F}^m \}$ / $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

Example.

$$\left[\begin{array}{ccc|c} 1 & 3 & 8 & 0 \\ 1 & 2 & 6 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

t

$\begin{bmatrix}-2t\\-2t\\t\end{bmatrix} = t\begin{bmatrix}-2\\-2\\1\end{bmatrix}$

$Null(A) = \text{Span}\left\{ \begin{bmatrix}-2\\-2\\1\end{bmatrix} \right\}$

$1 + 2t = 0$

Left null space of A.

$Null(A^T) = \{ \cancel{A^Ty} \; y \in \mathbb{R}^m : \underset{n \times m}{A^T} \underset{m \times n}{y} = 0 \} \to n \times 1$

$= \{ y \in \mathbb{R}^m : y^T A = 0 \} \text{ in } 1 \times n$

Example. $Null(A^T)$:

$$\begin{bmatrix} 1 & 3 & 8 \\ 1 & 2 & 6 \\ 0 & 1 & 2 \end{bmatrix}^T \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\overset{R^T}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 2 \end{bmatrix}} \overset{C^T}{\begin{bmatrix} 1 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 3 & 2 & 1 \\ 8 & 6 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

In $\mathbb{R}^{m\times n}$,

Remark: $A = \begin{bmatrix} u_1^T \\ \vdots \\ u_m^T \end{bmatrix}$

$AX = 0$ means.

X is orthogonal to the $u_1, \dots, u_m$ in $\mathbb{R}^n$

$$\boxed{\langle x, y\rangle = \left\langle \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \right\rangle = x_1y_1 + \cdots + x_ny_n}$$

and $\langle x, y\rangle = 0$ means / is called x and y are orthogonal.

i.e., X is orthogonal to $v_1, \dots, v_r$

where $\cancel{C(A^T)}$ $\mathrm{Row}(A) = C(A^T) = \mathrm{span}\{v_1, \dots, v_r\}$

$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ x & x \end{bmatrix} \begin{bmatrix} 0 & x \\ 0 & x \end{bmatrix}$

Matrix factorization.

$$\boxed{A = LU,} \quad A = QR, \quad A = RDR^{-1}S \quad A = Q\Lambda Q^T \text{ if } A \text{ is symmetric}, \quad A = U\Sigma V^T.$$

1. LU factorization for invertible $A \in M_n$.

   To solve $Ax = b$ for many different vectors $b \in \mathbb{F}^n$.

   $Ax = b_1, \quad Ax = b_2, \quad \ldots \quad Ax = b_\ell \ldots$

   Write $A = LU$ where L is lower triangular, and U as upper triangular.

   Then solve $\boxed{Ax = LUx = b \text{ by solving } Ly = b \text{ and } Ux = y.}$

   How to write $LU = A$? We will see this in Section 1.4

   One can do an exchange of rows by a permutation matrix P to get $\boxed{PA = LU}$ if A is invertible.

   $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} A_1^T \\ A_2^T \\ A_3^T \end{bmatrix} = \begin{bmatrix} A_2^T \\ A_3^T \\ A_1^T \end{bmatrix}$

   Question What if A is not invertible or A is not a square matrix?

   $m \times n$

2. QR Decomposition for $A \in M_{m,n}$ with $m \geq n$.

   Recall the inner product on $\mathbb{F}^n$ defined by $\langle x, y \rangle = y^* x = \sum_{j=1}^n x_j \bar{y}_j$, and the inner product norm $\|x\| = \langle x, x \rangle^{1/2}$.

   V^T real

   We are interested in orthonormal set/basis $\{v_1, \ldots, v_n\}$ such that $V^*V = I_n$ if $V = [v_1|\cdots|v_n]$, i.e., $\langle v_i, v_j \rangle = \delta_{ij}$.

   Complex $\quad Q^T$ real $\quad$ complex

   $Q^TQ = I_n$

   To solve $Ax = b$ with b in the column space of b, we can solve $Rx = Q^*Ax = Q^*b$.

3. $A = RDR^{-1} = \sum_{j=1}^n \lambda_j x_j y_j^T$ where R has columns $x_1, \ldots, x_n$ and R^{-1} has rows $y_1^T, \ldots, y_n^T$.

   Then $A^k = \sum_{j=1}^n \lambda_j^k x_j y_j^T$.

   $A = [R] \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_n \end{bmatrix} R^{-1}$

   If $H = H^*$ is Hermitian, then $H = Q\Lambda Q^* = \sum_{j=1}^n \lambda_j v_j v_j^*$ with $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ so that $H^k = \sum_{j=1}^n \lambda_j^k v_j v_j^*$.

4. $A = U\Sigma V^*$. Then $A = \sum_{j=1}^r s_j u_j v_j^*$.

<u>Remark</u> Solving $Ax = b$ by $x = A^{-1}b$ is not a good way

because A^{-1} may be bad, A is ill-conditioned.

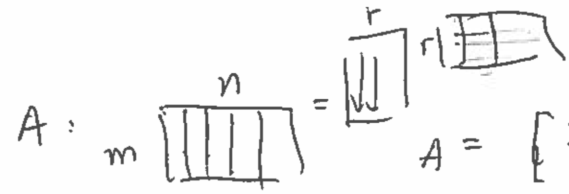
<u>Example</u> $\begin{bmatrix} 1 & 1 \\ 1 & 1.00000001 \end{bmatrix}$

$Dx = b$

$A = \begin{bmatrix} 1 & 0 \\ * & 1 \end{bmatrix} \begin{bmatrix} x & * \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} d_1 & \\ 0 & d_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$

$\begin{bmatrix} 1 & 1 \\ 1 & 1.0\cdots01 \end{bmatrix}^{-1} \quad \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 2+0.01 \end{bmatrix}$



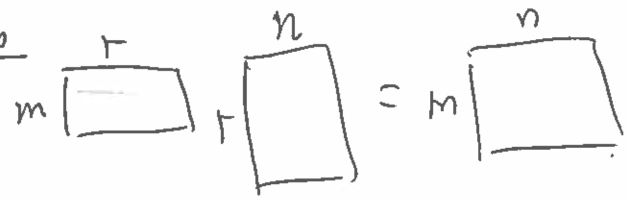
$A = \begin{bmatrix} 1 & 4 & 7 & 10 \\ 2 & 5 & 8 & 11 \\ 3 & 6 & 9 & 12 \end{bmatrix}$

I.3 The Four Fundamental Subspaces

Let $A \in M_{m,n}$. We have

$\boxed{\text{the column space } C(A) \text{ in } \mathbb{F}^m}$ contains all combination of columns of A,
the null space $N(A)$ contains x in $\mathbb{F}^n$ such that $Ax = 0$.
the row space $C(A^T)$ contains all combination of columns of A^T,
the left null space $N(A^T)$ contains y in $\mathbb{F}^m$ such that $A^T y = 0$.

Proposition Let $A \in M_{m,n}$. Then $C(A)$ and $C(A^T)$ have the same dimension r; $N(A)$ has dimension $n - r$, $N(A^T)$ has dimension $m - r$.



Theorem Let ~~$A \in M_{m,r}, B \in M_{r,n}$~~ A, B be matrices.

A is $m \times r$ B is $r \times n$

1. rank $(AB) \le \min\{\text{rank}(A), \text{rank}(B)\}$.

A, B are $m \times n$

2. rank $(A + B) \le$ rank (A) + rank (B).

A is $m \times n$

3. rank (A^*A) = rank (AA^*) = rank (A) = rank (A^*).

A is $m \times r$ B is $n \times r$

4. rank $(AB) = r$ if rank (A) = rank $(B) = r$.

$A B_1$

$= A \begin{bmatrix} b_{11} \\ \vdots \\ b_{rn} \end{bmatrix} = b_{11} u_1 + \cdots + b_{rn} u_m$

$A = [u_1 | \cdots | u_r]$

Proof: ① Col ~~space of~~ $AB = [AB_1 | \cdots | AB_n]$

$= \begin{bmatrix} A_1^T B \\ \vdots \\ A_m B \end{bmatrix}$

Col(AB) in the col(A)

$\boxed{\text{Row}(AB) = \text{Col}((AB)^T)}$

$\therefore \dim \text{Col}(AB) \le \dim \text{Col}(A)$.

$\therefore$ rank $(AB) \le$ rank(A).

Also Row$(AB) \subseteq$ Row(B)

$\therefore$ rank$(AB) \le$ rank(B).

$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 7 \\ 6 & 8 \end{bmatrix}$

$= \left[A \begin{bmatrix} 5 \\ 6 \end{bmatrix} \; A \begin{bmatrix} 7 \\ 8 \end{bmatrix} \right]$

$= \left[5 \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 6 \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right]$

② $\operatorname{col}(A+B) \subseteq \operatorname{Span}\{\operatorname{col}(A) \cup \operatorname{col}(B)\}$

Let

$A = [A_1 | \cdots | A_n] \quad B = [B_1 | \cdots | B_n]$

$A+B = [A_1+B_1 | \cdots | A_n+B_n]$

$\therefore \operatorname{col}(A+B) \subseteq \operatorname{Span}\{A_1, B_1, A_2, B_2, \ldots, A_n, B_n\}$

$\boxed{\subseteq \operatorname{col}[A_1 | B_1 | A_2 | \cdots A_n | B_1 \cdots | B_n]}$

$\dim \operatorname{col}[A_1 \cdots A_n | B_1 \cdots B_n] \leq \operatorname{rank}(A) + \operatorname{rank}(B)$ ////

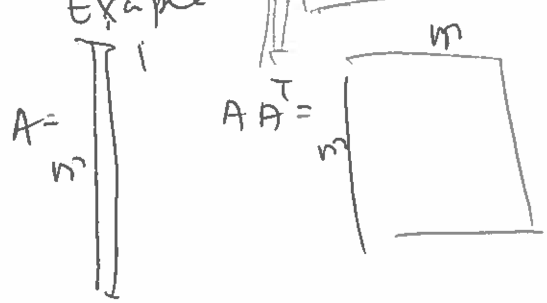
Remark: $\operatorname{rank}(A+B) = \operatorname{rank}(A) + \operatorname{rank}(B) - \dim(\operatorname{col}(A) \cap \operatorname{col}(B))$

---

③ A is $m \times n$ $\quad \operatorname{rank}(\overset{n\times n}{A^T A}) = \operatorname{rank}(\overset{m \times m}{A A^T})$

$= \operatorname{rank}(\underset{m\times n}{A}) = \operatorname{rank}(\underset{n\times m}{A^T})$

Exaple



$A^T A = [\ \cdot\ ]$

$\operatorname{rank}(A) + \dim \operatorname{null}(A) = n$

$\operatorname{rank}(A^T A) + \dim \operatorname{null}(A^T A) = n$

Claim: $\operatorname{null}(A) = \operatorname{null}(A^T A)$

$(\subseteq)$ $x \in \mathbb{R}^n$, $x \in \operatorname{null}(A)$, $\therefore Ax = 0$, $\therefore A^T A x = 0$ $\quad \therefore x \in \operatorname{null}(A^T A)$

$(\supseteq)$ $x \in \operatorname{null}(A^T A)$ $\quad A^T A x = 0$, $\quad x^T A^T A x = 0$

i.e., $Ax = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$ with $b_1^2 + \cdots + b_m^2 = 0$ $\quad (b_1 \cdots b_m) \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$

$b_1 = \cdots = b_m = 0$ $\quad \therefore x \in \operatorname{null}(A)$

(4) A is $m \times r$ B is $~~r~~ \times n$

rank$(AB) = r$ if and only if rank$(A) = r$ and
rank$(B) = r$

($\Rightarrow$) If. rank$(AB) = r$ Then column space $(AB) = r$
and row space $(AB) = r$.

~~Now S if ra~~

$r \times n$

~~Now~~ $C((AB)^T)$ $=$ Row$(AB) \subseteq$ Row(B), must have dim r

Col$(AB) \subseteq$ col(A), must have dim r.

$m \times r$

($\Leftarrow$) If rank$(A) = r$, rank$(B) = r$.
then

col$(AB) = r$. by the CR decomposition
interpretation

I.4 LU factorization

Recall that we want to write $A = LU$ for an invertible $A \in M_n$.

We can use elementary operations $E_r \cdots E_1[A|I_n] = [U|R]$, $\qquad R = E_r \cdots E_1$.

Then $RA = U$ so that $R = L^{-1} = E_1^{-1} \cdots E_r^{-1}$.

A better way: $A - u_1 v_1^T = [0] \oplus A_1$ if we let $u_1 = (a_{11}, \dots, a_{n1})^T / a_{11}$ and $v_1^T = (a_{11}, \dots, a_{1n})$.

Then use induction on A_1 to get $A = \sum_{j=1}^n u_j v_j^T$.

Example

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 2 & 7 & 8 \end{bmatrix} \xrightarrow{E_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 3 & 2 \end{bmatrix} \xrightarrow{E_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} = U.$$

i.e., $(E_2 E_1) A = U \quad \therefore A = E_1^{-1} E_2^{-1} U = LU$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & +3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix}.$$

Remark We can apply the procedure as long as the $(1,1)$ entry of A_k is nonzero in each step.

Else, one can apply a permutation P to A so that $PA = LU$.

Remark

$$\left[\begin{array}{c|c} 1 & 0 \\ \hline x & \begin{smallmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{smallmatrix} \end{array}\right] \left[\begin{array}{c|c} 1 & 0 \\ \hline -x & \begin{smallmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{smallmatrix} \end{array}\right] = \left[\begin{array}{c|c} 1 & \\ \hline \begin{smallmatrix} 0 \\ \vdots \\ 0 \end{smallmatrix} & I_{n-1} \end{array}\right]$$

$$\left[\begin{array}{c|c} 1 & 0 \cdots 0 \\ \hline x & \begin{smallmatrix} 1 & & \\ & \ddots & \\ 0 & & 1 \end{smallmatrix} \end{array}\right] \left[\begin{array}{c|c} 1 & 0 \cdots 0 \\ \hline \begin{smallmatrix} 0 \\ \vdots \\ 0 \end{smallmatrix} & \begin{smallmatrix} 1 & & 0 \\ & \ddots & \\ C_{ij} & & 1 \end{smallmatrix} \end{array}\right] = \left[\begin{array}{c|c} 1 & 0 \\ \hline x & \begin{smallmatrix} 1 & & \\ & \ddots & \\ C_{ij} & & 1 \end{smallmatrix} \end{array}\right]$$

$$A - \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \left[\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & 5 & 7 \\ 0 & 7 & 8 \end{array}\right]$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 5 & 7 \\ 0 & 7 & 8 \end{pmatrix} - \begin{bmatrix} 0 \\ 5/5 \\ 7/5 \end{bmatrix} \begin{bmatrix} 0 & 5 & 7 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

$$A - \begin{bmatrix} 1 \\ x \\ x \end{bmatrix} [\ \ ] - \begin{bmatrix} 0 \\ 1 \\ x \end{bmatrix} [\ \ ] - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 8 \end{bmatrix}$$