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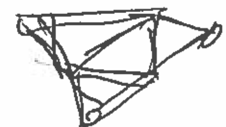
MATH309: Intermediate Linear Algebra
Homework 1

$m \times n \quad n \times 1$

$A x = A y \in \mathbb{R}^m$



5 **2.** $z_1 = 0$, $z_2 = x - y$. For $z_1 = 0$, it's obvious that $Az_1 = 0$.

Assume $A = [u_1|\cdots|u_n]$, $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, $y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$. Since $Ax = Ay$, $u_1x_1 + \cdots + u_nx_n = u_1y_1 + \cdots + u_ny_n$. So $u_1(x_1 - y_1) + \cdots + u_n(x_n - y_n) = 0$, which means $A(x - y) = Az_2 = 0$.

5 **6.** Since these are corners of parallelograms, the sum of two corners must be equal to the sum of the other two. So the three possible corners are: $x_1 = (1,1) + (4,2) - (1,3) = (4,0)$, $x_2 = (1,1) - (4,2) + (1,3) = (-2,2)$, $x_3 = -(1,1) + (4,2) + (1,3) = (4,4)$.

5 **9.** Since the column space is all of $\mathbb{R}_3$ 3, m has to be 3. So the rank should be $\leqslant 3$, but there has to be 3 linearly independent columns to span the whole $\mathbb{R}_3$ 3, so $r = 3$, and $n \geqslant 3$.

$M = 3$

10. $C_1 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$, $C_2 = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{bmatrix}$.

10 **11.**

$$A_1 = \begin{bmatrix} 1 & 3 & -2 \\ 3 & 9 & -6 \\ 2 & 6 & -4 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 3 & -2 \end{bmatrix} = C_1R_1.$$

$$A_2 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix} = C_2R_2.$$

14. A counterexample would prove the first two.

0 Let $A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 6 & 8 \\ 6 & 9 & 12 \end{bmatrix}$. They have the same column space, but the row space is different. Also, they have different basic columns. However, their ranks must be the same because if they are not, it's impossible for A and B to have the same column space.

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 2 & 3 & 4 \\ 4 & 6 & 8 \\ 6 & 9 & 12 \end{bmatrix}$$

I.4 LU factorization

Recall that we want to write $A = LU$ for an invertible $A \in M_n$. How to do it?

- Use elementary operations by subtracting multiple of upper rows from lower rows to get an upper triangular matrix $E_r \cdots E_1 A = U$.
- Let $R = E_r \cdots E_1$. Then R is lower triangular, and $A = R^{-1}U$ so that $L = E_1^{-1} \cdots E_r^{-1}$.
- A better way: $A - u_1 v_1^T = [0] \oplus A_1$ if we let $u_1 = (a_{11}, \dots, a_{n1})^T / a_{11}$ and $v_1^T = (a_{11}, \dots, a_{1n})$.
- Then use induction on A_1 to get $A = \sum_{j=1}^{n} u_j v_j^T$.

Example

$$\begin{bmatrix}3&4&9\\2&7&10\\3&8&17\end{bmatrix} - \begin{bmatrix}3&4&9\\2&0&0\\3&0&0\end{bmatrix} = \begin{bmatrix}0&0&0\\0& & \\0& & \end{bmatrix}$$

$$\begin{bmatrix}3&4&9\\2& & \\3& & \end{bmatrix} - \begin{bmatrix}1\\2/3\\3/3\end{bmatrix}\begin{bmatrix}3&4&9\end{bmatrix} - \begin{bmatrix}0\\ \\ \end{bmatrix}\begin{bmatrix}0& & \end{bmatrix} = \begin{bmatrix}0&0&0\\0&0&0\\0&0& \end{bmatrix}$$

Remark We can apply the procedure as long as the $(1,1)$ entry of A_k is nonzero in each step.
Else, one can apply a permutation P to A so that $PA = LU$.
Just choose P so that the leading $k \times k$ submatrix is invertible for each $k = 1, \dots, n$.

$$\begin{bmatrix}3&4&9\\2& & \\3& & \end{bmatrix} = \underbrace{\begin{bmatrix}1\\2/3\\3/3\end{bmatrix}}_{u_1}\underbrace{\begin{bmatrix}3&4&9\end{bmatrix}}_{v_1^T} + \underbrace{\begin{bmatrix}0\\ \\ \end{bmatrix}}_{u_2}\underbrace{\begin{bmatrix}0& & \end{bmatrix}}_{v_2^T} + \underbrace{\begin{bmatrix}0\\0\\ \end{bmatrix}}_{u_3}\underbrace{\begin{bmatrix}0&0& \end{bmatrix}}_{v_3^T}$$

$$= \begin{bmatrix}1&0&0\\2/3&1&0\\3/3& &1\end{bmatrix}\begin{bmatrix}3&4&9\\0& & \\0&0& \end{bmatrix}.$$

$$\begin{bmatrix}u_1 & u_2 & \cdots & u_n\end{bmatrix}\begin{pmatrix}v_1^T\\ \vdots \\ v_n^T\end{pmatrix} = u_1 v_1^T + \cdots + u_n v_n^T$$

$Ax = b$.

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ \cancel{x}_{21} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & \cancel{x}_{12} \\ 0 & x_{22} \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$m \times n \to A$

$$\begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} \begin{bmatrix} a_{ij} \end{bmatrix} =$$

exchanging r & s rows of A

In general, we can apply a pemution matrix P to A so ~~to~~ as to permute the rows of A.

Resulting matrix : PA

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$P_1 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} [1\ 0\ 0] - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$[1\ \ 1]$

$\therefore P_1 A \neq LU$.

Choose

$$P = \overset{P_2}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}} \overset{P_1}{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$PAx = Pb_1$

$PAx = Pb_{10000}$

$PAx = LUx = \hat{b}_i$

Let $A \in M_n$ be invertible

Theorem: There is P permutation such that PA has invertible $k \times k$ leading principal submatrix for $k = 1, \dots, n$.

& $PA = LU$

Proof: Let $A \in M_n$ first $\exists\ a_{j1} \neq 0$. $j \in \{1, \dots, n\}$

Else $\operatorname{rank}(A) \leq n-1$

So
$$P_1 A = \begin{bmatrix} a_{11} & a_{12} & \cdots \\ * & & \\ \vdots & & \\ * & & \end{bmatrix} \qquad a_{11} \neq 0.$$

$$= [u_1\ u_2\ \cdots\ u_n]$$

By Gaussian / Elementary row operation

$$P_1 A \sim \begin{bmatrix} 1 & & \\ 0 & 1 & \\ 0 & 0 & 1 \end{bmatrix}$$

corresponds to some rows, say row r_1

Then
$$P_2 P_1 A = \begin{array}{c} \\ r \to \\ \\ \\ \end{array} \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{r1} & \cdots & a_{rn} \\ a_{21} & \cdots & \\ a_{n1} & \cdots & \end{bmatrix} \qquad (a_{r1}, \quad)$$

will have leading 2×2 submatrix invertible.

Repeating this argument, ~~k×k~~ there is
or by induction
a permutation matrix

$P_{n-1} \cdots P_2 P_1 = P$ will produce PA such that the leading $k \times k$ submatrix is invertible. $k = 1, \dots, n$.

So $PA = LU$

Remark: If there are different choices for P so that $PA = LU$ to avoid small # ... to avoid P s.t. LUX

Recall : $\underset{n\times n}{A}\,\underset{n\times 1}{x} = \underset{n\times 1}{b_i}$ $\quad i = 1, \cdots, N$

$A = QR$, Q is orthogonal, R upper triangular

$Q^TQ = QQ^T = I$

Then $Ax = b$
$\quad \| $
QRx

$\therefore$ $Rx = Q^Tb$ ✓✓

$\left(\,\boxed{\;}\,\right) x = Q^Tb$

$A = \left[\,\vdots \;\vdots\; \cdots\,\right]$

$A^{-1}b$

I.5 Orthogonality

- Inner product, norm, and orthogonal vectors.

Let $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^n$

The inner product of x, y, denoted by $\langle x, y\rangle = x_1y_1 + \cdots + x_ny_n \in \mathbb{R}$
$= y^Tx$

Define $\|x\| = \langle x, x\rangle^{1/2}$
$= \left(\sum x_i^2\right)^{1/2}$

- Pythagoras theorem. $\|x - y\|^2 = \|x\|^2 + \|y\|^2$.

Define $\langle x, y\rangle = \|x\|\|y\|\cos\theta$

x, y are orthogonal if $\cos\theta = 0$
i.e., $\langle x, y\rangle = 0$

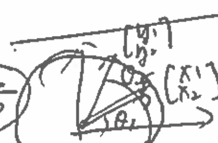
- Cosine law. $\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2\|x\|\|y\|\cos\theta$, where $\langle x, y\rangle = \|x\|\|y\|\cos\theta$

Example:

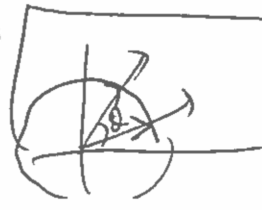
$$x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad y = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

$$\langle x, y\rangle = -1 < 0$$

Properties of $\langle x, y\rangle$

1. $\langle \lambda x, y\rangle = \lambda\langle x, y\rangle$
2. $\langle x, y\rangle = \langle y, x\rangle$
3. $\langle x, x\rangle \geq 0$ & equality holds if and only if $x = (0, 0, \cdots, 0)^T$
4. $\langle x_1 + x_2, y\rangle = \langle x_1, y\rangle + \langle x_2, y\rangle$
5. 

$x_1y_1 + x_2y_2$
$= \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2$
$= \cos(\theta_1 - \theta_2)$



$\langle x, y\rangle = \|x\|\left\langle \frac{x}{\|x\|}, \frac{y}{\|y\|}\right\rangle \|y\|$
$= \|x\|\|y\|\cos\theta$

Example

$$x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad y_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \quad y_2 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

$$\langle x, y_1 \rangle = 0 \qquad \langle x, y_2 \rangle = 0$$

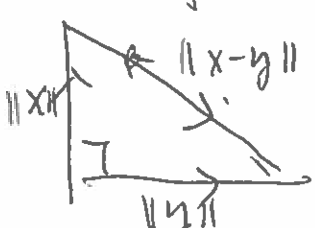
$$\begin{array}{cc} & 3 \\ 1 & \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \end{array} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Example

$$x_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad x_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Pythagoras Theorem



$$\|x\|^2 + \|y\|^2 = \|x-y\|^2$$

$$\Leftrightarrow \quad \langle x, y \rangle = 0$$

Proof: $\|x-y\|^2 = \langle x-y, x-y \rangle = \langle x, x \rangle - 2\langle x, y \rangle + \langle y, y \rangle$

$$= \|x\|^2 - 2\|x\|\,\|y\| \cos\theta + \|y\|^2$$

$\therefore \quad \|x-y\|^2 = \|x\|^2 + \|y\|^2 \Leftrightarrow \|x\| \cos\theta = 0 \Leftrightarrow \langle x, y \rangle = 0$

Example:

$$x = \begin{bmatrix} 1 \\ 3 \\ 4 \\ 5 \end{bmatrix}, \quad y = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\langle x, y \rangle = 1 + 0 + 4 + 0 = 5$$

$$\cos\theta = \frac{5}{\|x\| \, \|y\|} = \frac{5}{\sqrt{1+9+16+25}\,\sqrt{1+1}} = \frac{5}{\sqrt{51}\,\sqrt{2}}$$

Note that $|\langle x, y \rangle| \leq \|x\| \, \|y\| \quad \forall x, y \in \mathbb{R}^n$

Cauchy Schwarz inequality

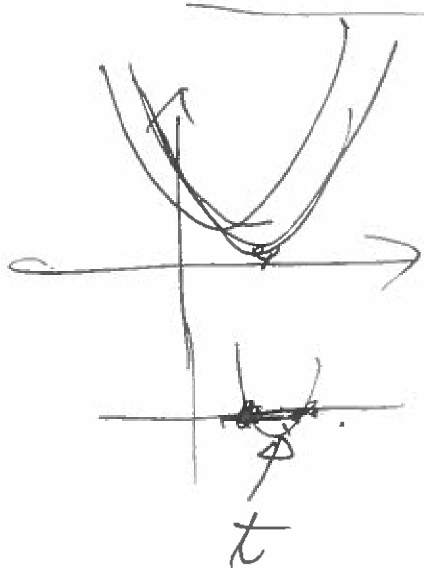
$\cancel{0 \leq \underbrace{\langle x - ty, x - ty \rangle}_{\|x-ty\|^2}}$

for any $t \in \mathbb{R}$

Proof: $0 \leq \|x - ty\|^2 = \langle x - ty, x - ty \rangle$

$$= \langle x, x \rangle - 2t \langle x, y \rangle + \langle y, y \rangle t^2$$

$$= c - 2t(b) + a t^2$$



∴ ~~zero~~ roots of the equations $\dfrac{2b \pm \sqrt{4b^2 - 4ac}}{2}$ will never give two distinct real number if and only if $4b^2 - 4ac \leq 0$

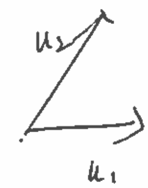
$$\therefore \quad ac \geq b^2$$

$$\|x\|^2 \|y\|^2 \geq |\langle x, y \rangle|^2.$$

Gram-Schmidt process and QR factorization

- Let $\{u_1, \ldots, u_k\} \subseteq \mathbb{R}^n$ be a linearly independent set. We can an orthonormal set $\{v_1, \ldots, v_k\}$ by the Gram-Schmidt process as follows.

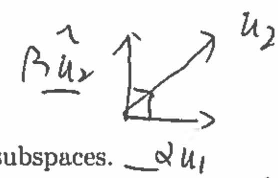
  Let $v_1 = u_1$. If $v_1, \ldots, v_\ell$ are constructed and $\ell < k$, set



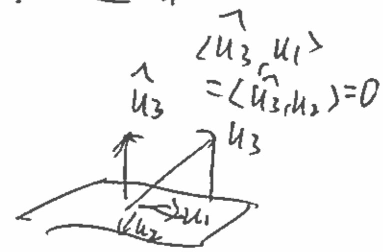
$$\tilde{v}_{\ell+1} = u_{\ell+1} - a_1 v_1 - \cdots a_\ell v_\ell) \quad \text{and} \quad v_{\ell+1} = \tilde{v}_{\ell+1}/\|\tilde{v}_{\ell+1}\|$$

  with $a_j = \langle v_j, u_{\ell+1} \rangle$ for $j = 1, \ldots, \ell$.

- As a result, we can construct an orthonormal basis for a subspace.



- Orthogonal subspaces. For example, row space and null space are orthogonal subspaces.



- Also, we have the QR factorization.

$$[u_1 | \cdots | u_k] = [v_1 | \cdots | v_k] R = QR$$

  for an upper triangular matrix $R \in M_k$.

Example