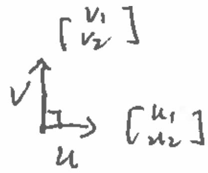


$\langle u, v\rangle = v^T u = u_1v_1 + u_2v_2$

$= 0 \iff u, v$ are orthogonal

$\|u\|\|v\|\cos\theta = \langle u, v\rangle$

Gram-Schmidt process and QR factorization

- Let $\{u_1, \ldots, u_k\} \subseteq \mathbb{R}^n$ be a linearly independent set. We can an orthonormal set $\{v_1, \ldots, v_k\}$ by the Gram-Schmidt process as follows.

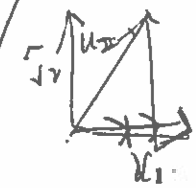
  Let $v_1 = \frac{u_1}{\|u_1\|}$. If $v_1, \ldots, v_\ell$ are constructed and $\ell < k$, set

  $$\tilde{v}_{\ell+1} = \boxed{u_{\ell+1}} - (a_1 v_1 + \cdots + a_\ell v_\ell) \quad \text{and} \quad v_{\ell+1} = \tilde{v}_{\ell+1}/\|\tilde{v}_{\ell+1}\|$$

  with $a_j = \langle v_j, u_{\ell+1}\rangle$ for $j = 1, \ldots, \ell$.

$\langle v_i, v_j\rangle = 0$, $i \neq j$

$\langle v_i, v_i\rangle = 1$ $\forall i$



- As a result, we can construct an orthonormal basis for a subspace.

- Orthogonal subspaces. For example, row space and null space are orthogonal subspaces.

$\langle \hat{u}_3, u_1\rangle = \langle \hat{u}_3, u_2\rangle = 0$

- Also, we have the QR factorization.

  $$[u_1|\cdots|u_k] = [v_1|\cdots|v_k]R = QR$$

  for an upper triangular matrix $R \in M_k$.

Example $\quad u_1 = \begin{bmatrix}1\\1\\0\end{bmatrix}, \quad u_2 = \begin{bmatrix}1\\1\\1\end{bmatrix}, \quad u_3 = \begin{bmatrix}1\\2\\3\end{bmatrix}$

Want:
$u_2 = \alpha_1 v_1 + \hat{v}_2$
$\hat{v}_2 = u_2 - \alpha_1 v_1$

$v_1 = \dfrac{\begin{bmatrix}1\\1\\0\end{bmatrix}}{\sqrt{2}} \qquad \hat{v}_2 = u_2 - \alpha_1 v_1$

s.t.

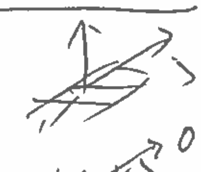
$0 = \langle \hat{v}_2, v_1\rangle = \langle u_2 - \alpha_1 v_1, v_1\rangle = \langle u_2, v_1\rangle - \alpha\langle v_1, v_1\rangle$
$= \langle u_2, v_1\rangle - \alpha_1$

$\therefore \alpha_1 = \left\langle \begin{bmatrix}1\\1\\1\end{bmatrix}, \dfrac{\begin{bmatrix}1\\1\\0\end{bmatrix}}{\sqrt{2}} \right\rangle = \dfrac{2}{\sqrt{2}}$

$\hat{v}_2 = \begin{bmatrix}1\\1\\1\end{bmatrix} - \dfrac{2}{\sqrt{2}} \dfrac{\begin{bmatrix}1\\1\\0\end{bmatrix}}{\sqrt{2}} = \begin{bmatrix}1\\1\\1\end{bmatrix} - \begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\1\end{bmatrix}, \quad v_2 = \dfrac{\hat{v}_2}{\|\hat{v}_2\|} = \begin{bmatrix}0\\0\\1\end{bmatrix}$

$\hat{v}_1 = u_1$
$\hat{v}_2 + \alpha_1 \hat{v}_1 = u_2 - \alpha_1 \hat{v}_1$
$0 = \langle \hat{v}_2, \hat{v}_1\rangle = \langle \hat{v}_2, u_2\rangle - \alpha_1\langle \hat{v}_1, \hat{v}_1\rangle$
$\Rightarrow \alpha \sim - \langle \hat{v}_1, u_2\rangle$

Want:
$u_3 = \hat{v}_3 + \alpha_1 v_1 + \alpha_2 v_2$
$\hat{v}_3 = u_3 - \alpha_1 v_1 - \alpha_2 v_2$
$0 = \langle \hat{v}_3, v_1\rangle = \langle u_3, v_1\rangle - \alpha\cancel{\langle v_1, v_1\rangle} - \alpha_2\cancel{\langle v_2, v_1\rangle}$
$\alpha_1 = \langle u_3, v_1\rangle$



Example (cont'd) $u_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. $v_1 = \dfrac{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}{\sqrt{2}}$ $v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$\hat{v}_3 = u_3 - \alpha_1 v_1 - \alpha_2 v_2$$

$$= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \frac{3}{\sqrt{2}} \frac{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}{\sqrt{2}} - 3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \frac{3}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - 3\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

$$\therefore \quad v_3 = \frac{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}{\sqrt{2}}$$

For $\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix}$

Check

$$\begin{bmatrix} v_1^T \\ v_2^T \\ v_3^T \end{bmatrix} \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Remark: $\{v_1 \cdots v_k\} \subseteq \mathbb{R}^n$ is an orthonormal set

$$\Longleftrightarrow \begin{bmatrix} v_1^T \\ \vdots \\ v_k^T \end{bmatrix} \begin{bmatrix} v_1 & \cdots & v_k \end{bmatrix} = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \in M_k$$

Remark:

① $\text{col}[u_1 \cdots u_j] = \text{col}[v_1 \cdots v_j]$

In fact,

$$[u_1 \cdots u_j] = \underset{Q}{[v_1 \cdots v_j]} \underset{R}{\begin{bmatrix} r_{11} & r_{12} & & \\ 0 & r_{22} & & \\ \vdots & 0 & \ddots & \\ 0 & & 0 & \beta_j \end{bmatrix}} \quad j \times j$$

$$u_1 = r_{11} v_1$$

$$u_2 = r_{12} v_1 + r_{22} v_2 \qquad Q^T Q = I_j$$

$$u_3 = \cdots$$

②

$$\text{col}[u_1 \cdots u_k] = \text{col}[v_1 \cdots v_k]$$

∴ the subspace in $\mathbb{R}^n$ generated by $\{u_1 \cdots u_k\}$ is the same as that generated by $\{v_1, \cdots, v_k\}$.

So $\{v_1, \cdots, v_k\}$ is an orthonormal basis.

In general every vector v in the subspace can be written as

$$v = \underline{\alpha_1} u_1 + \cdots + \underline{\alpha_k} u_k \quad \Big| \quad [u_1 \cdots u_k]\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_k \end{bmatrix} = v$$

① $A = QR$

② $Ax = b$

③ $QRx = b$

④ $Rx = Q^T b$

Also $v = \beta_1 v_1 + \cdots + \beta_k v_k + 0$

$$\beta_j = \langle v, v_j \rangle, \; j = 1, \cdots, k$$

$$[v_1 \cdots v_k]\begin{bmatrix} \beta_1 \\ \vdots \\ \beta_k \end{bmatrix} = v$$

∴

$$\begin{bmatrix} v_1^T \\ \vdots \\ v_k^T \end{bmatrix}[v_1 \cdots v_k]\begin{bmatrix} \beta_1 \\ \vdots \\ \beta_k \end{bmatrix} = \begin{bmatrix} v_1^T v \\ v_2^T v \\ \vdots \\ v_k^T v \end{bmatrix}$$

$$= I_k \begin{bmatrix} \beta_1 \\ \vdots \end{bmatrix}$$

$$\begin{bmatrix} \beta_1 \\ \vdots \end{bmatrix}$$

Remark: For $\mathbb{C}^n$, $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$

$$\langle x, y \rangle = \bar{y}_1 x_1 + \cdots + \bar{y}_n x_n \in \mathbb{C}$$

Example $\left\langle \begin{pmatrix} 1+i \\ 1 \end{pmatrix} \middle| \begin{pmatrix} i \\ 2 \end{pmatrix} \right\rangle$

$$= (-i, 2) \begin{pmatrix} 1+i \\ 1 \end{pmatrix} = -i + 1 + 2 = -i + 3.$$

Still define $|\langle x, y \rangle| = \|x\| \|y\| \cos\theta$

Gram Schmidt works! ${}^{n}A^{k} = QR$, $Q^* Q = I_k$

R is $k \times k$ upper triangular

Example $u_1 = \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix}$, $u_2 = \begin{bmatrix} 1 \\ i \\ 1 \end{bmatrix}$, $u_3 = \begin{bmatrix} 1 \\ 2 \\ 3i \end{bmatrix}$

$v_1 = \dfrac{\begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix}}{\sqrt{2}}$ $\hat{v}_2 = u_2 - \alpha_1 v_1$, $\alpha_1 = \left\langle \begin{bmatrix} 1 \\ i \\ 1 \end{bmatrix}, \dfrac{\begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix}}{\sqrt{2}} \right\rangle$

$= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. $v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\hat{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 3i \end{bmatrix} - \alpha_1 v_1 - \alpha_2 v_2$ $\alpha_1 = \langle u_3, v_1 \rangle$

$\alpha_2 = \langle u_3, v_2 \rangle$

$= (1 + \frac{i}{2}) \begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix}$ $v_3 = \dfrac{\begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix}}{\sqrt{2}}$ //

If $Q = [v_1\ v_2\ v_3]$

then $[u_1\ u_2\ u_3] = Q \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{bmatrix}$, $Q^* Q = I_3$

$Q^* = (\overline{Q^T}) = (\bar{Q})^T$

Special classes of matrices

$Q^T Q$

- Matrix $Q \in M_{n,m}$ with orthonormal columns. $Q^*Q = I_m$. Then QQ^* is a projection.

Example:

$P_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} / \sqrt{2} \; [1 \; 1 \; 0]/\sqrt{2} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$P_2 = \begin{bmatrix} 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 \\ 0 & 1 \end{bmatrix} [\quad] = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$[v_1 \; v_2] \begin{bmatrix} v_1^T \\ v_2^T \end{bmatrix} = v_1 v_1^T + v_2 v_2^T$

$P_3 = [v_1 v_2 v_3] \begin{bmatrix} v_1^T \\ v_2^T \\ v_3^T \end{bmatrix} = QQ^T = I_3$

- A matrix $Q \in M_n$ such that $Q^*Q = I_n$ is a unitary/orthogonal matrix.

  The set of unitary / orthogonal matrices form a group.

$Q^T Q = I$

$\Leftrightarrow QQ^T = I$

QQ^*

$v_3 v_3^T = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} / \sqrt{2} \; [-1 \; 1 \; 0]/\sqrt{2} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$

- Hadamard matrices.

  If $n = 4k$, then there is a Hadamard matrix in in $M_n(\mathbb{R})$.

  The smallest unknown case: 668.

$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ $\quad AA^T = A^TA = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ $\quad \frac{A}{\sqrt{2}} \cdot \frac{A^T}{\sqrt{2}} = \frac{AA^T}{2} = I$

$A \otimes A = \begin{bmatrix} A & A \\ A & -A \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$

Kronecker product

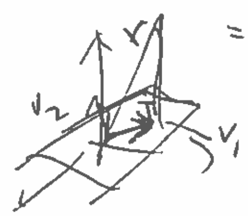
$(A \otimes A) \otimes A$, 8×8 $\quad 2^k$

$(A \otimes A)(A^T \otimes A^T) = I_4$

$= (AA^T) \otimes (AA^T)$

$= 2I_2 \otimes 2I_2$

$[v_1 v_2] \begin{bmatrix} v_1^T \\ v_2^T \end{bmatrix} x = [v_1 v_2] \begin{bmatrix} v_1^T x \\ v_2^T x \end{bmatrix} = \alpha_1 v_1 + \alpha_2 v_2$



- Hadamard matrices are useful in quantum computing.

$A \otimes B = [a_{ij}] \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ \vdots & & & \\ a_{m1}B & \cdots & \cdots & a_{mn}B \end{bmatrix}$

$n = 4$

- Householder reflection has the form $I - 2uu^*$ for a unit vector. The matrix has eigenvalues $-1, 1, \ldots, 1$.

$\underset{m\times n}{}\;\underset{p\times q}{}\;\underset{n\times r}{}\;\underset{q\times s}{}$

$(A \otimes B)(C \otimes D) = (A \cdot C) \otimes (BD)$

$(mp \times nq)\;(nq \times rs)$

$mp \times rs = X = a_1 (A_1 \otimes B_1)(C_1 \otimes D_1) + a_2 (A_2 \otimes B_2)(C_2 \otimes D_2) + \cdots = s_1 [\,] + s_2 [\,] + \cdots s_k$

n

m

Example: $u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ $u_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $u_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Remark $u_1 = \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix}$ $\begin{bmatrix} 1 \\ i \\ 1 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 2 \\ 3i \end{bmatrix}$

$$\langle x, y \rangle = x_1 \bar{y}_1 + x_2 \bar{y}_2 + x_3 \bar{y}_3$$

$v_1 = \dfrac{\begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix}}{\sqrt{2}}$

$$\hat{v}_2 = u_2 - \langle u_2, v_1 \rangle v_1$$

$$= u_2 - \frac{(1\ -i\ 0)\begin{pmatrix} 1 \\ i \\ 1 \end{pmatrix}}{\sqrt{2}} \frac{\begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix}}{\sqrt{2}}$$

$$= u_2 - \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\hat{v}_3 = u_3 - \langle u_3, v_1 \rangle v_1 - \langle u_3, v_2 \rangle v_2$$

$$= \begin{bmatrix} 1 \\ 2 \\ 3i \end{bmatrix} - \frac{(1, -i, 0)\begin{pmatrix} 1 \\ 2 \\ 3i \end{pmatrix}}{\sqrt{2}} \frac{\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}}{\sqrt{2}} - (0\ 0\ -1)\begin{pmatrix} 1 \\ 2 \\ 3i \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 2 \\ 3i \end{pmatrix} - \frac{1-2i}{2}\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} - (3i)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \left(1 + \frac{i}{2}\right)\begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}$$

$\begin{matrix} 1 & -1 \\ i & -i \end{matrix}$

$2 - \frac{i}{2}$

$1 - \frac{1}{2} + \frac{2i}{2}$ $\quad 1 - \frac{i}{2}$ $\quad 2 - i$ $\quad i$

$$v_3 = \frac{\begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}}{\sqrt{2}}$$