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MATH309: Intermediate Linear Algebra
Homework 2

Extra Credit Question: Example $A \in M_2(\mathbb{R})$, $\det(A)=0$, $\mathrm{row}(A)=\mathrm{col}(A)$

5 **I.2 #7** $A = \begin{bmatrix}1 & 4\\2 & 5\\3 & 6\end{bmatrix}$, $B = \begin{bmatrix}1 & 2\\0 & 0\end{bmatrix}$, $AB = \begin{bmatrix}1 & 2\\2 & 4\\3 & 6\end{bmatrix}$. ✓

5 **I.3 #4** We are not sure whether A is symmetric. For example, $A = \begin{bmatrix}1 & 3\\2 & 4\end{bmatrix}$ fulfills all the requirements, but it's not symmetric. ✓

5 **I.3 #7** They don't always have the same null space. For instance, $A = \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}$, $A^2 = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$. $x = \begin{bmatrix}1\\1\end{bmatrix}$ is in A^2's null space but not in A's null space.

Can we conclude that $A = A^T$?

$B = A[I|I|I]$, $B = [A|A|A]$ $\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix} = A(v_1+v_2+v_3) = 0$

5 **I.3 #10** Assume that A is a $m \times n$ matrix. Then B is a $m \times 3n$ matrix. Let v_1, v_2, v_3 be three $n \times 1$ matrix. Then the vector $V = \begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}$ will be in the null space of B if $v_1 + v_2 + v_3 = 0$. ✓

4 ~~5~~ **I.4 #3** $\begin{bmatrix}2 & 1 & 0\\0 & 4 & 2\\6 & 3 & 5\end{bmatrix} - \begin{bmatrix}1\\0\\3\end{bmatrix}\begin{bmatrix}2 & 1 & 0\end{bmatrix} = \begin{bmatrix}0 & 0 & 0\\0 & 4 & 2\\0 & 0 & 5\end{bmatrix}$. $\begin{bmatrix}0 & 0 & 0\\0 & 4 & 2\\0 & 0 & 5\end{bmatrix} - \begin{bmatrix}0 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}0 & 0 & 0\\0 & 4 & 2\\0 & 0 & 0\end{bmatrix} = \begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 5\end{bmatrix}$.

Thus, $A = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\3 & 0 & 1\end{bmatrix}\begin{bmatrix}2 & 1 & 0\\0 & 4 & 2\\0 & 0 & 5\end{bmatrix}$.

$E = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\-3 & 0 & 1\end{bmatrix}$ ⇒ It's asking for E.

5 **I.4 #5** (1) d has to be 0 because $1 \cdot d + 0 \cdot 0 = 0$. However, according to u_{21} of A, $d \cdot l + 1 \cdot 0 = 2$, which means that d can't be 0, contradiction.
(2) According to entries of the matrix A, we can easily derive that d=1, e=1, l=1 and f=0. So $U = \begin{bmatrix}1 & 1 & g\\0 & 0 & h\\0 & 0 & i\end{bmatrix}$. According to u_{31} of A, m=1. Then according to u_{32} of A, $1 \cdot 1 + n \cdot 0 + 1 \cdot 0 = 2$, which is impossible. Thus, LU decomposition is impossible. ✓

5 **I.4 #6** c=2 would make the second pivot position 0, so row exchange is needed. c=1 would make the third pivot position 0, so it's impossible to do LU decomposition. ✓

5 **I.4 #7** We need four non-zero pivots, so $a \neq 0$, and $a \neq b, b \neq c, c \neq d$. ✓

5 **I.4 #10** A_k factors into $L_k U_k$ because the upper left corner of L and U are derived from A_k.

Please show more work

5 **I.4 #11** To permute $\begin{bmatrix}1 & 3\\2 & 4\end{bmatrix}$ in advance, we need to exchange the rows and then exchange the columns. To do that, we can use $\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}1 & 3\\2 & 4\end{bmatrix}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$, which gives us $\begin{bmatrix}4 & 2\\3 & 1\end{bmatrix}$. Then,

$$A - \begin{bmatrix}1\\\frac{3}{4}\end{bmatrix}\begin{bmatrix}4 & 2\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & -\frac{1}{2}\end{bmatrix}. \text{ So } \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}1 & 3\\2 & 4\end{bmatrix}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} = \begin{bmatrix}1 & 0\\\frac{3}{4} & 1\end{bmatrix}\begin{bmatrix}4 & 2\\0 & -\frac{1}{2}\end{bmatrix}$$

✓

Special classes of matrices

- Matrix $Q \in M_{n,m}$ with orthonormal columns $Q^*Q = I_m$. Then QQ^* is a projection.

$(AB)^T(AB) = (B^TA^T)(AB) = B^TIB = I_n$

$\Rightarrow C^TC = I_n$

$\Rightarrow C = AB$ is orthogonal

A, B orthogonal matrices $A^TA = I$, $B^TB = I$

- A matrix $Q \in M_n$ such that $Q^*Q = I_n$ is a unitary/orthogonal matrix.

  The set of unitary / orthogonal matrices form a group.

$QQ^T = Q^TQ = I_n$

(G0) $A, B \in G$, $AB \in G$
(G1) $(AB)C = A(BC)$
(G2) I
(G3) $A^{-1}A = AA^{-1} = I$

- Hadamard matrices.

  If $n = 4k$, then there is a Hadamard matrix in in $M_n(\mathbb{R})$.

  The smallest unknown case: 668. $= 4 \times 167$

$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

$H = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$(H \otimes H) \otimes H \otimes \cdots \otimes H$ (k) — $2^k \times 2^k$ Hadamard matrix

- Hadamard matrices are useful in quantum computing.

$I - 2uu^T$

$(I - uu^T)^T = I^T - (uu^T)^T = I - (u^T)^Tu^T = I - uu^T$

- Householder reflection has the form $I - 2uu^*$ for a unit vector. The matrix has eigenvalues $-1, 1, \ldots, 1$.

In $\mathbb{R}^3$: $u^Tx = 0$

$u = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} / \sqrt{1^2+2^2+3^2+4^2+5^2}$

Claim: $I - 2uu^T$ is orthogonal

$(I - uu^T)u = -u$

$(I - vv^T)v = v$ if $u^Tv = 0$

$(I - 2uu^T)^T(I - 2uu^T) = (I - 2uu^T)(I - 2uu^T) = I - 2uu^T - 2uu^T + 4uu^Tuu^T = I - 4uu^T + 4uu^T = I$

$(I - 2uu^T)u = I \cdot u - 2uu^Tu = u - 2u = -u$

$v \in \mathrm{null}(u^T)$ = orthogonal complement of u

$(I - uu^T)v = Iv - uu^Tv = v - 0 = v$

$$Ax = \lambda x, \; x \neq 0$$ ($= \lambda I x$)

i.e., $(A-\lambda I)x$

Example: $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \in M_2(\mathbb{R})$

$\det(\lambda I - A) = \lambda^2 + 1 = 0$ has no real solution.

Over $\mathbb{C}$, $\lambda^2+1 = (\lambda-i)(\lambda+i)$ $\therefore$ Complex eigenvalues are $i, -i$.

Over real, no eigenvalues & eigenvectors

Over complex we can find $x_1 \neq 0$, $x_2 \neq 0$ s.t. $Ax_1 = ix_1$, $Ax_2 = (-i)x_2$

$A[x_1 x_2] = [Ax_1 \mid Ax_2] = [ix_1 \mid (-i)x_2] = [x_1 \mid x_2]\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$

$[x_1 \mid x_2]^{-1} A [x_1 \mid x_2] = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$

I.6 Eigenvalues, eigenvectors

Recall. We solve $Ax = \lambda x$ by solving (1) $\det(\lambda I - A) = 0$ for λ, and (2) $(\lambda I - A)x = 0$.

Theorem There is an invertible S such that $S^{-1}AS = D = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ if and only if A has n linearly independent eigenvectors (the columns of S).

Proof. ($\Rightarrow$) If $S^{-1}AS = \begin{bmatrix} \lambda_1 & & 0 \\ 0 & & \lambda_n \end{bmatrix}$, then $AS = S\begin{bmatrix} \lambda_1 & & 0 \\ 0 & & \lambda_n \end{bmatrix}$ ..

Let $S = [x_1 | \cdots | x_n]$. Then $A[x_1 \cdots x_n] = [x_1 \cdots x_n]\begin{pmatrix} \lambda_1 & & 0 \\ 0 & & \lambda_n \end{pmatrix}$.

Then $\{x_1, \ldots, x_n\}$ is linearly independent because col(S) has dim n. & $Ax_i = \lambda_i x_i$, $i = 1, \ldots, n$.

$\therefore$ A has n linearly independent eigenvectors $x_1, \ldots, x_n$.

($\Leftarrow$) Suppose A has n linearly independent eigenvectors $x_1, \ldots, x_n$. Then $Ax_i = \lambda_i x_i$ for some eigenvalues $\lambda_1, \ldots, \lambda_n$. Let $S = [x_1 | \cdots | x_n]$. Then $AS = [Ax_1 | \cdots | Ax_n] = [\lambda_1 x_1 | \cdots | \lambda_n x_n] = [x_1 \cdots x_n]\begin{bmatrix} \lambda_1 & & 0 \\ 0 & & \lambda_n \end{bmatrix} = S\begin{bmatrix} \lambda_1 & & \\ & & \lambda_n \end{bmatrix}$

$\therefore S^{-1}AS = \begin{bmatrix} \lambda_1 & & 0 \\ 0 & & \lambda_n \end{bmatrix}$

Example Find S such that $S^{-1}AS = \operatorname{diag}(d_1, d_2)$ for $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.

$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

$\det(\lambda I - A) = \det\begin{bmatrix} \lambda-1 & -1 \\ -1 & \lambda \end{bmatrix} = \lambda(\lambda-1) - 1 = \lambda^2 - \lambda - 1$.

$\therefore \lambda = \dfrac{1 \pm \sqrt{1+4}}{2} = \dfrac{1 \pm \sqrt{5}}{2}$

Solve $[A - \lambda I]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ for $\lambda = \frac{1 \pm \sqrt{5}}{2}$

to get $S = [x_1\, x_2] = \begin{bmatrix} 1 & \frac{\sqrt{5}-1}{2} \\ \frac{\sqrt{5}-1}{2} & -1 \end{bmatrix}$.

So that $S^{-1}AS = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix}$

Alternatively, if $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is diagonalizable, then there is an invertible S s.t. $S^{-1}AS = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

$\therefore A = S\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}S^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Contradiction.

Remark:

① For real matrices, there may not be real values

② For complex matrices, there are always eigenvalues & each distinct eigenvalue has at least one "independent" eigenvector.

③ But there may not be n linearly independent eigenvectors for $A \in M_n(\mathbb{C})$

Example $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

0 is the only eigenvalue & $\text{null}(A - 0I)$ has dim 1

$Ax = \lambda x$

$A(bx) = bAx = b\lambda x = \lambda(bx)$

Applications

Recursive relation. Suppose $a_1, \ldots, a_k$ are given and $a_n = d_1 a_{n-1} + \cdots + d_k a_{n-k}$. We can find formula for a_n as follows.

Write $x_n = (a_n, \ldots, a_{n-k+1})^T$ and $A = \begin{pmatrix} d_1 & d_2 & \cdots & d_k \\ 1 & 0 & \cdots & 0 \\ & \ddots & \ddots & \vdots \\ & & 1 & 0 \end{pmatrix} \in M_k$.

$$\begin{bmatrix} a_n \\ \vdots \\ a_{n-k} \end{bmatrix} = \begin{bmatrix} d_1 \cdots d_k \\ 1 \quad 0 \\ \ddots \end{bmatrix} \begin{bmatrix} a_{n-1} \\ \vdots \\ a_{n-k-1} \end{bmatrix} = [\quad]^{(n-k)} \begin{bmatrix} a_k \\ \vdots \\ a_1 \end{bmatrix}$$

Then $x_n = Ax_{n-1} = A^2 x_{n-2} = \cdots = A^{n-k} x_k$.

If $A = SDS^{-1} = \lambda_1 S(:,1)S^{-1}(1,:) + \lambda_k S(:,k)S^{-1}(k,:)$, where $D = \text{diag}(\lambda_1, \ldots, \lambda_k)$, then

$$x_n = SD^kS^{-1} = \lambda_1 S(:,1)S^{-1}(1,:) + \lambda_k S(:,k)S^{-1}(k,:)x_k$$

so that $a_n = e_1^T SD^kS^{-1}x_k$.

Example Suppose $f_0 = 0, f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$. Then $x_n = A^{n-1}x_1$ with $x_1 = (1,0)^T$ and $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = SDS^{-1}$ with $D = \text{diag}(\lambda_1, \lambda_2) = \text{diag}((1+\sqrt{5})/2, (1-\sqrt{5})/2)$ and

$$S = \begin{bmatrix} 1 & (\sqrt{5}-1)/2 \\ (\sqrt{5}-1)/2 & -1 \end{bmatrix} \quad \text{and} \quad S^{-1} = \frac{2}{5-\sqrt{5}} S.$$

Thus, $a_n = \frac{1}{\sqrt{5}}(\lambda_1^n + \lambda_2^n)$. $\lambda_1 = \frac{1+\sqrt{5}}{2}$ $\lambda_2 = \frac{1-\sqrt{5}}{2}$

Exercise 1 Determine a formula for a_n if $f_0 = 1, f_1 = 2$ and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$.

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} f_n \\ f_{n-1} \end{bmatrix} = \begin{bmatrix} f_{n+1} \\ f_n \end{bmatrix}$$

$$\begin{bmatrix} f_{n+1} \\ f_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} f_n \\ f_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} f_{n-1} \\ f_{n-2} \end{bmatrix} = \cdots =$$

$$\begin{bmatrix} f_n \\ f_{n-1} \end{bmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{bmatrix} f_{n-1} \\ f_{n-2} \end{bmatrix} = A^2 \begin{bmatrix} f_{n-2} \\ f_{n-3} \end{bmatrix} = A^{n-1} \begin{bmatrix} f_1 \\ f_0 \end{bmatrix}$$

$$= \left(S \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} S^{-1}\right)^{n-1} \begin{bmatrix} f_1 \\ f_0 \end{bmatrix}$$

Recall $A = S \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix} S^{-1}$

$$= \left(S \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} S^{-1}\right) \cdots \left(S \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix} S^{-1}\right) \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$S = [x_1\ x_2]$, $S^{-1} = \begin{bmatrix} y_1^T \\ y_2^T \end{bmatrix}$

$$= S \begin{bmatrix} \lambda_1^{n-1} & 0 \\ 0 & \lambda_2^{n-1} \end{bmatrix} S^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Differential equations. Consider the system $Ax' = x$ with $x(0) = x_0$.

For one variable, $x' = ax$ with $x(0) = x_0$, we have $x = e^{at}x_0$.

Suppose $S^{-1}AS = D = \operatorname{diag}(d_1, \ldots, d_n)$. Then

$$A = SDS^{-1} = d_1 S(:,1)S^{-1}(1,:) + \cdots + d_n S(:,n)S^{-1}(n,:).$$

For $y = Sx$, $Sx' = y'$ and $Dy' = y$ so that

$$y = e^{Dt}y_0 = \begin{bmatrix} e^{id_1 t} & & \\ & \ddots & \\ & & e^{id_n t} \end{bmatrix} y_0 \quad \text{and} \quad x = S^{-1}y = S^{-1}e^{Dt}Sx_0.$$

That is,

$$x = (e^{d_1 t}S(:,1)S^{-1}(1,:) + \cdots + e^{d_n t}S(:,n)S^{-1}(n,:))x_0.$$

$x_1'(t) = a_{11}x_1(t) + \cdots + a_{1n}x_n(t)$

$x_n'(t) = a_{n1}x_1(t) + \cdots + a_{nn}x_n(t)$

$y_1'(t) = \lambda_1 y_1(t)$

$y_n'(t) = \lambda_n y_n(t)$

Example: Solve $x' = Ax$ with $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and $x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = SDS^{-1}$ with $D = \operatorname{diag}(\lambda_1, \lambda_2) = \operatorname{diag}((1+\sqrt{5})/2, (1-\sqrt{5})/2)$ and

$$S = \begin{bmatrix} 1 & (\sqrt{5}-1)/2 \\ (\sqrt{5}-1)/2 & -1 \end{bmatrix} \quad \text{and} \quad S^{-1} = \frac{2}{5-\sqrt{5}}S.$$

$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$

So,

$$x = \frac{1}{\sqrt{5}} \begin{bmatrix} \lambda_1 e^{\lambda_1 t} - \lambda_2 e^{\lambda_2 t} \\ e^{\lambda_1 t} - e^{\lambda_2 t} \end{bmatrix}.$$

$x_1'(t) = x_1(t) + x_2(t)$

$x_2'(t) = x_1(t)$

In fact, one can solve the equation $x'' = x' + x$ with $x(0) = (1,0)^T$.

Now, $\begin{pmatrix} x' \\ x \end{pmatrix}' = A \begin{pmatrix} x' \\ x \end{pmatrix}'$ so that $x = \frac{1}{\sqrt{5}}e^{\lambda_1 t} - e^{\lambda_2 t}$.

$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = S\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$

Exercise Solve the system $x' = Ax$ with the above A and $x_0 = (2,1)^T$.