① Please be reminded that Exam I will take place on Thur.

② It will cover material up to SVD. (discussion up to last lecture - Thursday lecture.)

③ There will be 5 questions

Question 1. Ask for some examples.

Questions 2/3 Computation. Homework type

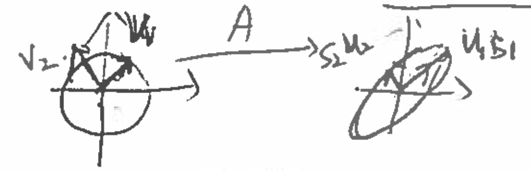
Questions 4/5 Short proofs

ⓐ such as SVD.

ⓑ Proofs asked in the homework

Applications of SVD

Transforming a circle to a ellipse Every $A : \mathbb{R}^2 \to \mathbb{R}^2$ will transform the unit circle to an ellipse.



Given A, ~~there exist~~ we have

$$A[v_1 v_2] = [u_1 \; u_2]\begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix}$$

Finding the maximum of $\|Ax\|$ among all unit vectors x. Use this method to compute

In the Hermitian/symmetric case: $\boxed{x^* A^* A x}$

$$Av_1 = s_1 u_1$$

~~Remark: For~~ For Hermitian matrix $A = A^*$, max $x^*Ax = \lambda_1$, min $x^*Ax = \lambda_n$

Variation principle Finding max $\|Ax\|/\|x\|$ $\boxed{\text{subject to } v_1^*}$ where v_1 is a unit eigenvector for the eigenvalue s_1^2 of A^*A; we will get

$$Av_2 = s_2 v_2$$

Remark 6

$$A = U\begin{bmatrix} 3 & & 0 \\ & 2 & \\ 0 & & -4 \end{bmatrix}U^*$$

$(s_1, s_2, s_3) = (4, 3, 2)$

Approximation of a matrix $A = \sum_{j=1}^{k} s_j u_j v_j^*$ by $\tilde{A} = \sum_{j=1}^{\ell} s_j u_j v_j^*$.

$[a,b]$, ~~$f(t)$~~ $y = \alpha x + b$, $f(y)$

Application in function spaces We can find a basis of differentiable periodic functions on $[0, 2\pi]$ consisting of $\{\cos kt : k = 0, 1, 2,\} \cup \{\sin kt : k = 1, 2, \dots\}$. The inner product to two functions f, g equals $\boxed{\int_0^{2\pi} f(t)g(t)dt}$. Then we can approximate $f(t)$ by $\sum_{j=0}^{N}(a_j \cos jt + b_j \sin jt)$.

Recall that $\mathbb{P}[x] = \{a_0 + a_1x + a_2x^2 + \cdots + a_nx^n : a_0, \dots, a_n \in \mathbb{R}, n \in \{0, 1, \dots\}\}$ has a basis $\{1, x, x^2, \dots\}$

Periodic functions on $[0, 2\pi]$ can be approximated by $f(t) \approx \sum^N a_j \cos jt + \sum^N b_j \sin jt$

Finite difference In the study of the discrete form of a derivative, we consider $f(x) - f(x - \Delta x)$.

Let $D = \begin{pmatrix} 1 & & \\ -1 & 1 & \\ & -1 & 1 \\ & & -1 \end{pmatrix}$ with $D^T = \begin{pmatrix} 1 & -1 & & \\ & 1 & -1 & \\ & & 1 & -1 \end{pmatrix}$. In computation, it is useful important find DD^T and D^TD; see p.66

$f(x) - f(x - \Delta x)$

$$A^T \begin{pmatrix} f(t_1) \\ \vdots \\ \ \end{pmatrix} = \boxed{A^TA} \begin{pmatrix} c \\ \vdots \\ \ \end{pmatrix}$$

Remarks (SVD)

(1) Can write
$$A = \underset{\substack{m\times m\\ \text{unitary}}}{\hat{U}} \underset{m\times n}{\left[\begin{array}{c|c} \begin{matrix} s_1 & & \\ & \ddots & \\ & & s_k \end{matrix} & 0 \\ \hline 0 & 0 \end{array}\right]} \underset{n\times n \text{ unitary}}{\hat{V}^*}$$

(2)
$$A = \underset{m\times k}{\begin{bmatrix} U \end{bmatrix}} \underset{k\times k}{\begin{pmatrix} s_1 & & 0 \\ & \ddots & \\ 0 & & s_k \end{pmatrix}} \underset{k\times n}{\begin{pmatrix} V^* \end{pmatrix}}$$

Col(A) has an orthonormal basis $\{u_1, \dots, u_k\}$

Col(A^*) = row space of A

has an orthonormal basis $\{v_1, \dots, v_k\}$

(3) HOW to find s_1, v_1, u_1 ?

Note.
$$A^*A\hat{V} = \hat{V}\begin{bmatrix} s_1^2 & & 0 \\ & \ddots & \\ 0 & & s_n^2 \end{bmatrix} \qquad s_1^2 \geq \dots \geq s_n^2$$

Therefore $\hat{v}_1^*(A^*A)\hat{v}_1 = s_1^2 \qquad \|v_1\| = 1$

In general, for any unit vector y,
$$\|Ay\|^2 = \underline{y^* A^*} \underline{Ay} = \underline{y^* \hat{V}} \begin{bmatrix} s_1^2 & & 0 \\ & \ddots & \\ 0 & & s_n^2 \end{bmatrix} \hat{V}^* y$$
$$= \begin{bmatrix} \bar{u}_1 & \cdots & \bar{u}_n \end{bmatrix} \begin{bmatrix} s_1^2 & & 0 \\ & \ddots & \\ 0 & & s_n^2 \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$$
$$= \underline{s_1^2}|u_1|^2 + \dots + s_n^2|u_n|^2$$
$$\leq s_1^2|u_1|^2 + s_1^2|u_2|^2 + \dots + s_1^2|u_n|^2 = s_1^2(|u_1|^2 + \dots + |u_n|^2)$$
$$= s_1^2.$$

$\therefore$ ~~Comp~~

~~finding~~

$$s_1 = \max \{ \| Ay \| : \|y\| = 1 \}$$

Then $Av_1 = s_1 u_1)$. Then $A = s_1 u_1 v_1^* + A_2$.

Then Apply the same argument to A_2.

and find $\max \{ \|A_2 y\| : \|y\| = 1, y^* v_1 \}$

Then we get $A_2 v_2 = s_2 u_2$.

$\therefore$ $A = s_1 u_1 v_1^* + s_2 u_2 v_2^* + A_3$

Example:

$$A = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 4 & 5 & 6 \end{bmatrix}$$

$$= \underbrace{\frac{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}{\sqrt{5}}}_{U} \sqrt{5}\ \sqrt{77}\ \underbrace{\frac{\begin{bmatrix} 4 & 5 & 6 \end{bmatrix}}{\sqrt{16+25+36}}}_{V^T}$$

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$\sqrt{5 \times 77}$ ← s_1

To find SVD

Example: $A = \begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix}$

$$A^T A = \begin{bmatrix} 3 & 4 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 25 & 20 \\ 20 & 25 \end{bmatrix} = 5 \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$$

$$\det\left(\lambda I - \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}\right) = (\lambda-5)^2 - 16 = \lambda^2 - 10\lambda + 25 - 16$$

$$= \lambda^2 - 10\lambda + 9 = (\lambda - 9)(\lambda - 1).$$

$\therefore$ e.v. of $\begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$ is 9 & 1.

e.v. of $5\begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$ is $\underline{45}$ and $\underline{5}$. $(s_1^2 \quad s_2^2)$

Solve $\begin{bmatrix} 25 & 20 \\ 20 & 25 \end{bmatrix} - (45)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} / \sqrt{2}$

$\begin{bmatrix} 25 & 20 \\ 20 & 25 \end{bmatrix} - 5\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} / \sqrt{2}$

Singular values $\sqrt{45}$ $\sqrt{5}$

$$A\,\frac{\begin{bmatrix}1\\1\end{bmatrix}}{\sqrt{2}} = \frac{\begin{bmatrix}3&0\\4&5\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}}{\sqrt{2}} = \frac{\begin{bmatrix}3\\9\end{bmatrix}}{\sqrt{2}} = \frac{3\begin{bmatrix}1\\3\end{bmatrix}}{\sqrt{2}}$$

$$= \sqrt{45}\cdot\begin{bmatrix}1\\3\end{bmatrix}\Big/\sqrt{10}$$

$$A\,\frac{\begin{bmatrix}1\\-1\end{bmatrix}}{\sqrt{2}} = \frac{\begin{bmatrix}3&0\\4&5\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix}}{\sqrt{2}}$$

$$= \frac{\begin{bmatrix}3\\-1\end{bmatrix}}{\sqrt{2}} = \sqrt{5}\left(\begin{bmatrix}3\\-1\end{bmatrix}\Big/\sqrt{10}\right)$$

$$\frac{3}{\sqrt{2}} = \frac{\sqrt{45}}{\sqrt{10}} \quad \frac{\sqrt{9}}{\sqrt{2}}$$

$$\frac{\sqrt{5}}{\sqrt{10}} = \frac{1}{\sqrt{2}}$$

$$\therefore\ A\,\underbrace{\frac{\begin{bmatrix}1&1\\1&-1\end{bmatrix}}{\sqrt{2}}}_{V} = \underbrace{\frac{\begin{bmatrix}1&3\\3&-1\end{bmatrix}}{\sqrt{10}}}_{U}\ \underbrace{\begin{bmatrix}\sqrt{45}&0\\0&\sqrt{5}\end{bmatrix}}_{\begin{bmatrix}s_1&0\\0&s_2\end{bmatrix}}$$

$$\therefore\ A = \underbrace{\frac{\begin{bmatrix}1&3\\3&-1\end{bmatrix}}{\sqrt{10}}}_{U}\ \underbrace{\begin{bmatrix}\sqrt{45}&0\\0&\sqrt{5}\end{bmatrix}}_{\Sigma}\ \underbrace{\frac{\begin{bmatrix}1&1\\1&-1\end{bmatrix}}{\sqrt{2}}}_{V}$$

$n \times n$

$A[v_1\ v_2 \cdots v_k | v_{k+1} \cdots v_n]$

$= [s_1 u_1 | s_2 u_2 | \cdots | s_k u_k \cdots]$

$= [u_1 \cdots u_k \cdots u_n]\begin{bmatrix} s_1 & & \\ & \ddots s_k & 0 \\ & 0 & 0\end{bmatrix}$

1.8 Singular value decomposition, polar decomposition.

Theorem Let $A \in M_{m,n}$. There exists unitary $U \in M_{m,k}$ and $V \in M_{n,k}$ each with orthonormal columns such that $U^*U = I_k = V^*V$ and $\Sigma = \operatorname{diag}(s_1, \ldots, s_k)$ with $s_1 \geq \cdots \geq s_k > 0$ such that

$$\underbrace{A = \underset{m \times k}{U}\underset{k \times k}{\Sigma}\underset{k \times n}{V^*}} = \sum_{j=1}^{k} s_j u_j v_j^*.$$

$\hat{U} \begin{bmatrix} s_1 & & \\ & \ddots s_k & 0 \\ 0 & & 0\end{bmatrix}$

Proof. If $\tilde{V}^*A^*A\tilde{V} = \operatorname{diag}(s_1^2, \ldots, s_n^2)$, then the first k columns of $\tilde{V}$ are orthogonal columns of lengths $s_1, \ldots, s_k$ corresponding to the nonzero eigenvalues $s_1^2, \ldots, s_k^2$. So, there is a unitary $\tilde{U} \in M_m$ such that $A\tilde{V} = \tilde{U}\tilde{D}$, where $\tilde{D} \in M_{m,n}$ with (j,j) entry equal to s_j for $j = 1, \ldots, k$, and all other entries zero. So, $A = \tilde{U}\tilde{D}\tilde{V}^* = \sum_{j=1}^{k} s_j u_j v_j^*$, where $u_1, \ldots, u_k$ are the first k columns of $\tilde{U}$ and $v_1, \ldots, v_k$ are the first k columns of V. □

Note: $A = \hat{U} \begin{bmatrix} s_1 \ddots s_k & 0 \\ 0 & 0 \end{bmatrix} \hat{V}^*$ ⟵ $n \times n$ unitary

Definition The values $s_1 \geq \cdots \geq s_k$ are known as the (nonzero) singular values of A. The produce $U\Sigma V^*$ is called the singular decomposition of A. → $m \times m$ unitary

Note that AA^* and A^*A have eigenvalues $s_1^2 \geq \cdots \geq s_k^*$ and zoeros.

z, θ, $|z|$

Also, the Wielandt matrix $S = \begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix}$ has eigenvalues $s_1, \ldots, s_k, -s_k, \ldots, -s_1$ and zeros.

Theorem Let $A \in M_n$. Then $\boxed{A = UP = QV}$ for some unitary U, V and positive semidefinite P, Q with eigenvalues $s_1 \geq \cdots \geq s_n$.

Remark 1×1. $z \in \mathbb{C}$, $z = |z| e^{i\theta} = e^{i\theta}|z|$

Proof: $A = \underset{n \times n}{\hat{U}}\Sigma\underset{n \times n}{\hat{V}^*} = U(\hat{V}^*\hat{V})\Sigma\hat{V}^* = UP$, $U = \hat{U}\hat{V}^*$, PSD

$\Sigma = \begin{bmatrix} s_1 \ddots s_k & 0 \\ 0 & 0\end{bmatrix}$

Also $A = \hat{U}\Sigma\hat{V}^* = \hat{U}\Sigma\hat{U}^*(\hat{U}\hat{V}^*) = QV$

Example If A is rank one, then ...

Let $A = \begin{pmatrix} 3 & 0 \\ 4 & 5 \end{pmatrix}$. Then $A^TA = \begin{pmatrix} 25 & 20 \\ 20 & 25 \end{pmatrix}$ has eigenvalues $45, 5$ and eigenvectors $(1,1)^T$ and $(1,-1)^T$. So, $AV = U\operatorname{diag}(45, 5)$ with $u_1 = \sqrt{45}(1,3)^T/\sqrt{10}$ and $u_2 = (-3,1)^T/\sqrt{10}$.

The polar decomposition is:

$A = U \begin{bmatrix} \sqrt{45} & 0 \\ 0 & \sqrt{5} \end{bmatrix} V^T = U \begin{bmatrix} \sqrt{45} & 0 \\ 0 & \sqrt{5} \end{bmatrix} U^T (UV^T) = (UV^T) V \begin{bmatrix} \sqrt{45} & \\ & \sqrt{5} \end{bmatrix} V$

$U = \frac{1}{\sqrt{10}}\begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix}$, $V = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, P, $\frac{1}{\sqrt{10}}\begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, Q

<u>Remark</u> If $A = A^*$ has eigenvalues

$\lambda_1, \ldots, \lambda_n$.

then the singular values are

$|\lambda_1|, \ldots, |\lambda_n|$ arranged in descending order.

If $A = U \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} U^*$ $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$.

then the singular values are

$|\lambda_1|, \ldots, |\lambda_n|$. arranged in descending order

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If $A = \begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix}$, then. s.v. are

$\sqrt{45}, \sqrt{5}$

If $A = \begin{bmatrix} 3 & 0 \\ 4 & 3+4i \end{bmatrix}$

then the singular values s_1, s_2 are the nonnegative square roots of A^*A. which

are $\sqrt{45}, \sqrt{5}$

If $A = UBV$. $(UBV)^*(UBV) = V^*B^*U^*UBV$
$= V^*B^*BV$

then A^*A & B^*B has the same eigenvalues. $\therefore$ A & B have the same singular values.

Example. $\begin{bmatrix} 3 & 0 \\ 4 & 3+4i \end{bmatrix}$ has the same singular value

$$\begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 4 & 3+4i \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{3-4i}{5} \end{bmatrix}$$

If A is unitary the A has singular values $1, \dots, 1$.

Proposition. $A \in M_n$ is unitary $\Leftrightarrow$ A has singular values $1, \dots, 1$.