Let BV be the set of basic variables for $A'\mathbf{x} = \mathbf{b}'$ and NBV be the set of nonbasic variables for $A'\mathbf{x} = \mathbf{b}'$.

Case 1 $A'\mathbf{x} = \mathbf{b}'$ contains at least one row of the form $[0 \quad 0 \quad \cdots \quad 0|c](c \neq 0)$. In this case, $A\mathbf{x} = \mathbf{b}$ has no solution.

Case 2 If Case 1 does not apply and NBV, the set of nonbasic variables, is empty, then $A\mathbf{x} = \mathbf{b}$ will have a unique solution.

Case 3 If Case 1 does not hold and NBV is nonempty, then $A\mathbf{x} = \mathbf{b}$ will have an infinite number of solutions.

Linear Independence, Linear Dependence, and the Rank of a Matrix

A set V of m-dimensional vectors is **linearly independent** if the only linear combination of vectors in V that equals $\mathbf{0}$ is the trivial linear combination. A set V of m-dimensional vectors is **linearly dependent** if there is a nontrivial linear combination of the vectors in V that adds to $\mathbf{0}$.

Let A be any $m \times n$ matrix, and denote the rows of A by $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_m$. Also define $R = \{\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_m\}$. The **rank** of A is the number of vectors in the largest linearly independent subset of R. To find the rank of a given matrix A, apply the Gauss–Jordan method to the matrix A. Let the final result be the matrix $\bar{A}$. Then rank A = rank $\bar{A}$ = number of nonzero rows in $\bar{A}$.

To determine if a set of vectors $V = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ is linearly dependent, form the matrix A whose ith row is $\mathbf{v}_i$. A will have m rows. If rank $A = m$, then V is a linearly independent set of vectors; if rank $A < m$, then V is a linearly dependent set of vectors.

Inverse of a Matrix

For a given square ($m \times m$) matrix A, if $AB = BA = I_m$, then B is the **inverse** of A (written $B = A^{-1}$). The Gauss–Jordan method for inverting an $m \times m$ matrix A to get A^{-1} is as follows:

Step 1 Write down the $m \times 2m$ matrix $A|I_m$.

Step 2 Use EROs to transform $A|I_m$ into $I_m|B$. This will only be possible if rank $A = m$. In this case, $B = A^{-1}$. If rank $A < m$, then A has no inverse.

Determinants

Associated with any square ($m \times m$) matrix A is a number called the **determinant** of A (written det A or $|A|$). For a 1×1 matrix, det $A = a_{11}$. For a 2×2 matrix, det $A = a_{11}a_{22} - a_{21}a_{12}$. For a general $m \times m$ matrix, we can find det A by repeated application of the following formula (valid for $i = 1, 2, \ldots, m$):

$$\det A = (-1)^{i+1}a_{i1}(\det A_{i1}) + (-1)^{i+2}a_{i2}(\det A_{i2}) + \cdots + (-1)^{i+m}a_{im}(\det A_{im})$$

Here A_{ij} is the ijth **minor** of A, which is the $(m-1) \times (m-1)$ matrix obtained from A after deleting the ith row and jth column of A.

REVIEW PROBLEMS

Group A

1 Find all solutions to the following linear system:

$$\begin{aligned} x_1 + x_2 \quad &= 2 \\ x_2 + x_3 &= 3 \\ x_1 + 2x_2 + x_3 &= 5 \end{aligned}$$

2 Find the inverse of the matrix $\begin{bmatrix} 0 & 3 \\ 2 & 1 \end{bmatrix}$.

3 Each year, 20% of all untenured State University faculty become tenured, 5% quit, and 75% remain untenured. Each year, 90% of all tenured S.U. faculty remain tenured and 10% quit. Let U_t be the number of untenured S.U. faculty at the beginning of year t, and T_t the tenured number. Use matrix multiplication to relate the vector $\begin{bmatrix} U_{t+1} \\ T_{t+1} \end{bmatrix}$ to the vector $\begin{bmatrix} U_t \\ T_t \end{bmatrix}$.

4 Use the Gauss–Jordan method to determine all solutions to the following linear system:

$$\begin{aligned} 2x_1 + 3x_2 &= 3 \\ x_1 + x_2 &= 1 \\ x_1 + 2x_2 &= 2 \end{aligned}$$

5 Find the inverse of the matrix $\begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix}$.

6 The grades of two students during their last semester at S.U. are shown in Table 2.

Courses 1 and 2 are four-credit courses, and courses 3 and 4 are three-credit courses. Let GPA$_i$ be the semester grade point average for student i. Use matrix multiplication to express the vector $\begin{bmatrix} \text{GPA}_1 \\ \text{GPA}_2 \end{bmatrix}$ in terms of the information given in the problem.

7 Use the Gauss–Jordan method to find all solutions to the following linear system:

$$\begin{aligned} 2x_1 + x_2 &= 3 \\ 3x_1 + x_2 &= 4 \\ x_1 - x_2 &= 0 \end{aligned}$$

8 Find the inverse of the matrix $\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$.

9 Let C_t = number of children in Indiana at the beginning of year t, and A_t = number of adults in Indiana at the beginning of year t. During any given year, 5% of all children become adults, and 1% of all children die. Also, during any given year, 3% of all adults die. Use matrix multiplication to express the vector $\begin{bmatrix} C_{t+1} \\ A_{t+1} \end{bmatrix}$ in terms of $\begin{bmatrix} C_t \\ A_t \end{bmatrix}$.

TABLE 2

Student	Course 1	2	3	4
1	3.6	3.8	2.6	3.4
2	2.7	3.1	2.9	3.6

10 Use the Gauss–Jordan method to find all solutions to the following linear equation system:

$$\begin{aligned} x_1 \quad - x_3 &= 4 \\ x_2 + x_3 &= 2 \\ x_1 + x_2 \quad &= 5 \end{aligned}$$

11 Use the Gauss–Jordan method to find the inverse of the matrix $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$.

12 During any given year, 10% of all rural residents move to the city, and 20% of all city residents move to a rural area (all other people stay put!). Let R_t be the number of rural residents at the beginning of year t, and C_t be the number of city residents at the beginning of year t. Use matrix multiplication to relate the vector $\begin{bmatrix} R_{t+1} \\ C_{t+1} \end{bmatrix}$ to the vector $\begin{bmatrix} R_t \\ C_t \end{bmatrix}$.

13 Determine whether the set $V = \{[1 \quad 2 \quad 1], [2 \quad 0 \quad 0]\}$ is a linearly independent set of vectors.

14 Determine whether the set $V = \{[1 \quad 0 \quad 0], [0 \quad 1 \quad 0], [-1 \quad -1 \quad 0]\}$ is a linearly independent set of vectors.

15 Let $A = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{bmatrix}$.

a For what values of a, b, c, and d will A^{-1} exist?

b If A^{-1} exists, then find it.

16 Show that the following linear system has an infinite number of solutions:

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 1 \end{bmatrix}$$

17 Before paying employee bonuses and state and federal taxes, a company earns profits of \$60,000. The company pays employees a bonus equal to 5% of after-tax profits. State tax is 5% of profits (after bonuses are paid). Finally, federal tax is 40% of profits (after bonuses and state tax are paid). Determine a linear equation system to find the amounts paid in bonuses, state tax, and federal tax.

18 Find the determinant of the matrix $A = \begin{bmatrix} 2 & 4 & 6 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

19 Show that any 2×2 matrix A that does not have an inverse will have det $A = 0$.