Math 323 Opertions Research I   Homework 1

Sample solution based on that of Tianrui Zhu

1. Solve: $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$.   Solution. Apply Gaussian elimination, we have

   $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1+t \\ 3-t \\ t \end{bmatrix}$, with $t \in \mathbb{R}$.

2. According to the conditions given, it is intuitive that $U_{t+1} = 0.75U_t$. The number of tenured professors consists of two parts: the number of tenured professor who did not quit: $T^1_{t+1} = 0.9T^1_t$ and the number of untenured professors promoted:$T^2_{t+1} = 0.2U_t$. So if we write in matrix form, we have the following:

$$\begin{bmatrix} U_{t+1} \\ T_{t+1} \end{bmatrix} = \begin{bmatrix} 0.75 & 0 \\ 0.2 & 0.9 \end{bmatrix} \begin{bmatrix} U_t \\ T_t \end{bmatrix}$$

3. We solve the following equation: $\left[\begin{array}{cc|c} 2 & 3 & 3 \\ 1 & 1 & 1 \\ 1 & 2 & 2 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 2 & 3 & 3 \\ 1 & 2 & 2 \\ 1 & 2 & 2 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 2 & 3 & 3 \\ 1 & 2 & 2 \\ 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 2 & 3 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]$.

   So we have:

$$\left[\begin{array}{cc|c} 2 & 3 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

4. We have the following:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & -2 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$$

   Hence, $A^{-1} = \begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

5. Consider the following system:

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 4 \end{bmatrix}.$$

By Gaussian elimination,

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 & 3 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 4 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 2 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 4 \\ 0 & 0 & 1 & 1 & 3 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 2 \\ 0 & -1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & 4 \\ 0 & 0 & 1 & 1 & 3 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 2 \\ 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 & 3 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 2 \\ 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

There are only 3 basic variables among the 4 variables; the system has infinite solutions.

6. (a) We have the system

$$\left[\begin{array}{ccccc|c} 1 & 2 & 2 & 1 & 0 & 8 \\ 3 & 4 & 1 & 0 & 1 & 7 \end{array}\right].$$

Setting three variables to zero, we have have the following possibilities:

(i) $\left[\begin{array}{cc|c} 1 & 0 & 8 \\ 0 & 1 & 7 \end{array}\right] \rightarrow \begin{bmatrix} x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \end{bmatrix}$ (ii) $\left[\begin{array}{cc|c} 1 & 0 & 8 \\ 3 & 1 & 7 \end{array}\right] \rightarrow \begin{bmatrix} x_1 \\ x_5 \end{bmatrix} = \begin{bmatrix} 8 \\ -17 \end{bmatrix}$

(iii) $\left[\begin{array}{cc|c} 1 & 2 & 8 \\ 3 & 4 & 7 \end{array}\right] \rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -9 \\ 8.5 \end{bmatrix}$ (iv) $\left[\begin{array}{cc|c} 2 & 2 & 8 \\ 4 & 1 & 7 \end{array}\right] \rightarrow \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

(v) $\left[\begin{array}{cc|c} 2 & 1 & 8 \\ 4 & 0 & 7 \end{array}\right] \rightarrow \begin{bmatrix} x_2 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{7}{4} \\ \frac{9}{2} \end{bmatrix}$ (vi) $\left[\begin{array}{cc|c} 2 & 1 & 8 \\ 1 & 0 & 7 \end{array}\right] \rightarrow \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 7 \\ -6 \end{bmatrix}$

(vii) $\left[\begin{array}{cc|c} 2 & 0 & 8 \\ 1 & 1 & 7 \end{array}\right] \rightarrow \begin{bmatrix} x_3 \\ x_5 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ (viii) $\left[\begin{array}{cc|c} 1 & 2 & 8 \\ 3 & 1 & 7 \end{array}\right] \rightarrow \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{6}{5} \\ \frac{17}{5} \end{bmatrix}$

(ix) $\left[\begin{array}{cc|c} 1 & 1 & 8 \\ 3 & 0 & 7 \end{array}\right] \rightarrow \begin{bmatrix} x_1 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{7}{3} \\ \frac{17}{3} \end{bmatrix}$ (x) $\left[\begin{array}{cc|c} 2 & 0 & 8 \\ 4 & 1 & 7 \end{array}\right] \rightarrow \begin{bmatrix} x_2 \\ x_5 \end{bmatrix} = \begin{bmatrix} 4 \\ -9 \end{bmatrix}$

7. The maximum is achieved when $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1.2 \\ 0 \\ 3.4 \\ 0 \\ 0 \end{bmatrix}$ which has a maximum of 16.2.

Extra credit problem. Solving the problem yields $(W, P) = (7142.9, 4285.7)$. We try $y = (7143, 4285)^T$ and $z = (7142, 4286)^T$ and see that they are both feasible where as $u = (7143, 4286)$ is infeasible. Now, y will yield a larger profit of \$4142.7. So, we take the integer solution $(W, P) = (7143, 4285)$.