Introduction to Optimization

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What is Operations Research? What is Management Science?

- World War II : British military leaders asked scientists and engineers to analyze several military problems
  - Deployment of radar
  - Management of convoy, bombing, antisubmarine, and mining operations.
- The result was called *Military Operations Research*, later Operations Research, and also Management Science.

What is Management Science (Operations Research)?

- Today: Operations Research and Management Science mean

  "the use of *mathematical models* in providing guidelines to managers for making effective decisions within the state of the current information, or in seeking further information if current knowledge is insufficient to reach a proper decision."

- c.f. Decision science, systems analysis, operational research, systems dynamics, operational analysis, engineering systems, systems engineering, and more.

Operations Research Over the Years

- 1947
  - Project Scoop (Scientific Computation of Optimum Programs) with George Dantzig and others. Developed the simplex method for linear programs.
- 1950's
  - Lots of excitement, mathematical developments, queuing theory, mathematical programming.
    cf. A.I. in the 1960's
- 1960's
  - More excitement, more development and grand plans.
    cf. A.I. in the 1980's.

Operations Research Over the Years

- 1970's
  - Disappointment, and a settling down. NP-completeness.
    More realistic expectations.
- 1980's
  - Widespread availability of personal computers.
    Increasingly easy access to data. Widespread willingness of managers to use models.
- 1990's
  - Improved use of O.R. systems.
    Further inroads of O.R. technology, e.g., optimization and simulation add-ins to spreadsheets, modeling languages, large scale optimization. More intermixing of A.I. and O.R.

Operations Research Now

- LOTS of opportunities for OR as a field
- Data, data, data
  - E-business data (click stream, purchases, other transactional data, E-mail and more)
  - The human genome project and its outgrowth
- Need for more automated decision making
- Need for increased coordination for efficient use of resources (Supply chain management)

Optimization Models

Problems in theory and applications can be formulated as optimization problems.

- Define an objective function(s).
- Set up the decision variables.
- Find all the constraints.
- Solve them using optimization techniques.

Optimization is Everywhere

- It is embedded in language, and part of the way we think.
  - Firms want to maximize value to shareholders
  - People want to make the best choices
  - We want the highest quality at the lowest price
  - When playing games, we want the best strategy
  - When we have too much to do, we want to optimize the use of our time
  - etc.

Mathematical Optimization is nearly everywhere

- Finance
- Marketing
- E-business
- Telecommunications
- Games
- Operations Management
- Production Planning
- Transportation Planning
- System Design

Look for it! You will see opportunities for its use.

Model building process

- Formulate the problem.
- Observe the system.
- Formulate a mathematical model of the problem.
- Verify the model and use the model for prediction.
- Select a suitable alternative.
- Present the results and conclusion of the study.
- Implement and evaluate recommendation.

Example. A project in finding good match between Freshmen and advisors.

Read Example 1.3 CITGO petroleum

(Optimizing Refinery operation, and the supply distribution marketing system.)

Example 1.4 San Francisco Police Department Patrol Scheduling System.

Example 1.5 GE Capital – Payment system to reduce delinquent accounts.

Goal of Math 323

- We discuss some general optimization techniques for static/deterministic models (vs. dynamic and stochastic models).
- Specifically, we will study linear programming methods for linear optimization problem using matrix techniques.
- We use the techniques to study network models, integer programing, and some other advanced topics.
- If time permits, we may discuss some other topics such as non-linear optimization, and game theory.
- We will apply the theory to real life models, use existing or develop new algorithms to solve problems.

Linear Programming (our first tool, and probably the most important one.)

- minimize or maximize a linear objective
- subject to linear equalities and inequalities

maximize $3x + 4y$

subject to $5x + 8y \leq 24$

$x, y \geq 0$

A *feasible solution* satisfies all of the constraints.

x = 1, y = 1 is feasible; x = 1, y = 3 is *infeasible*.

An *optimal solution* is the best feasible solution.

The optimal solution is x = 4.8, y = 0.

Terminology

- *Decision variables:* e.g., x and y.
  - In general, there are quantities you can control to improve your objective which should completely describe the set of decisions to be made.
- *Constraints:* e.g., $5x + 8y \le 24$, $x \ge 0$, $y \ge 0$
  - Limitations on the values of the decision variables.
- *Objective Function.* e.g., $3x + 4y$
  - Value measure used to rank alternatives
  - Seek to maximize or minimize this objective
  - examples: maximize profit, minimize cost

MSR Marketing Inc.

adapted from Frontline Systems

- **Need to choose ads to reach at least 1.5 million people**
  - **Minimize Cost**
  - **Upper bound on number of ads of each type**

	TV	Radio	Mail	Newspaper
Audience Size	50,000	25,000	20,000	15,000
Cost/Impression	$500	$200	$250	$125
Max # of ads	20	15	10	15

Formulating as a math model

1. The decisions are how many ads of each type to choose. Let x_1 be the number of TV ads selected. Let x_2, x_3, x_4 denote the number of radio, mail, and newspaper ads. These are the "decision variables."
2. What is the objective? Express the objective in terms of the decision variables.
3. What are the constraints? Express these in terms of the decision variables.
4. If you have time, try to find the best solution.

The MSR Marketing Problem

Minimize $500 x_1 + 200 x_2 + 250 x_3 + 125 x_4$

subject to $50 x_1 + 25 x_2 + 20 x_3 + 15 x_4 \geq 1{,}500$

$$0 \leq x_1 \leq 20$$

$$0 \leq x_2 \leq 15$$

$$0 \leq x_3 \leq 10$$

$$0 \leq x_4 \leq 15$$

MSR Marketing

Gemstone Tool Company
(Thanks to Rob Freund)

- Privately-held firm
- Consumer and industrial market for construction tools
- Headquartered in Seattle
- Manufacturing plants in the US, Canada, and Mexico.
- Simplifying assumptions, for purposes of illustration:
  - Winnipeg, Canada plant
  - Wrenches and pliers.
  - Made from steel
  - Injection molding machine
  - Assembly machine

Data for the GTC Problem

	Wrenches	Pliers	Available
Steel	1.5	1.0	15,000 pounds
Molding Machine	1.0	1.0	12,000 hrs
Assembly Machine	.4	.5	5,000 hrs
Demand Limit	8,000	10,000	
Contribution ($ per item)	$.40	$.30	

We want to determine the number of wrenches and pliers to produce given the available raw materials, machine hours and demand.

Practice

- Formulate the GTC problem as a linear program.

- Let P = number of pliers made
- Let W = number of wrenches made

Formulating the GTC Problem

Step 1: Determine Decision Variables

W = number of wrenches manufactured

P = number of pliers manufactured

Step 2: Determine Objective Function

Maximize Profit =

$$.4\,W + .3\,P$$

The Formulation Continued

Step 3: Determine Constraints

Steel:	$1.5W + P \leq 15{,}000$
Molding:	$W + P \leq 12{,}000$
Assembly:	$0.4W + 0.5P \leq 5{,}000$
Wrench Demand:	$0 \leq W \leq 8{,}000$
Plier Demand:	$0 \leq P \leq 10{,}000$

Try to solve this problem.

Addressing managerial problems: A management science framework

1. Determine the problem to be solved
2. Observe the system and gather data
3. <u>Formulate a mathematical model of the problem and any important subproblems</u>
4. Verify the model and use the model for prediction or analysis
5. <u>Select a suitable alternative</u>
6. Present the results to the organization
7. Implement and evaluate

Dealing with very large versions of the problem

- Suppose that there are 10,000 products and 100 raw materials and processes that lead to constraints.
- Old technique used: write a Fortran program that generates the linear program
- New technique used: write an “algebraic version of the model”

An Algebraic Formulation

- n = number of items that are manufactured
  - e.g., in the previous example, n = 2;

- m = number of resource constraints
  - e.g., m = 2, {molding, and assembly}

Represent data in the previous example, using letters that stand for the data. In practice, the data is drawn from a table, and the letters refer to the table, and the subscripts indicate the position in the table.

- p_j = unit profit from item j, e.g., p_1 = .3;
- d_j = maximum demand for item j; e.g.,
- x_j = number of units of item j manufactured

- b_i = amount of resource i available
- a_{ij} = amount of resource i used in making item j

An Algebraic formulation

Maximize

$$\sum_{j=1}^{n} p_j x_j$$

Subject to

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i \quad for\, i = 1 \text{ to } m$$

$$x_j \le d_j \quad for\ j = 1 \text{ to } n$$

$$x_j \ge 0 \quad for\ j = 1 \text{ to } n$$

Linear Programs

- A *linear function* is a function of the form:

$$f(x_1, x_2, \ldots, x_n) = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$
$$= \sum_{i=1 \text{ to } n} c_ix_i$$

E.g., $3x_1 + 4x_2 - 3x_4$.

- A mathematical program is a *linear program (LP)* if the objective is a linear function and the constraints are linear equalities or inequalities.

E.g. $3x_1 + 4x_2 - 3x_4 \geq 7$

$x_1 - 2x_5 = 7$

- Typically, an LP has non-negativity constraints.

A non-linear program is permitted to have a non-linear objective and constraints.

- maximize $f(x,y) = xy$
- subject to $x - y^2/2 \leq 10$

$3x - 4y \geq 2$

$x \geq 0,\ y \geq 0$

An integer program is a linear program plus constraints that some or all of the variables are integer valued.

- Maximize $3x_1 + 4x_2 - 3x_3$

$$3x_1 + 2x_2 - x_3 \geq 17$$

$$3x_2 - x_3 = 14$$

$x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$ and

x_1, x_2, x_3 are all integers

An Algebraic formulation with equality constraints

Max or min

$$\sum_{j=1}^{n} c_j x_j$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j = b_i \quad for\ i = 1\ to\ m$$

$$x_j \geq 0 \quad for\ j = 1\ to\ n$$

Overview of this Lecture

- Course Administration
- Background on Operations Research (Management Science) and Optimization
- Course Themes and Goals
- Linear Programming Examples
  - MSR Marketing
  - GTC

Summary

- Answered the question: *What is Operations Research & Management Science?* and provided some historical perspective.
- Introduced the terminology of linear programming
- Two Examples:
  1. MSR Marketing
  2. Gemstone Tool Company
     - Small (2-dimensional) Linear Program, non-obvious solution
- We will discuss this problem in detail in Chapters 3 – 6.