Solving linear systems

• We use matrix notation to record a linear system $Ax = b$, where

$$A = [a_{ij}] \text{ is } m \times n,\ x = [x_1, \ldots, x_n]^T \text{ is } n \times 1 \text{ with variables } x_1, \ldots, x_n, \text{ and } b = [b_1, \ldots, b_m]^T.$$

• We use Guassian-Jordan reduction on the augmented matrix $[A|b]$ to determine all the solutions.

• In particular, apply elementary row operations and reduce it to the row echelon form.

Examples

- Three possibilities: (a) no solution, (b) one solution, (c) infinitely many solution.

  In terms of the (reduced) row echelon form:

  (a) happens if there is a row of the form $[0, \ldots, 0, r]$ with $r \neq 0$ in the row echelon form;

  (b) happens if (a) does not hold and there are n basic variables (corresponding to leading ones);

  (c) happens if (a) does not hold and there are non-basic variables.

Examples:

Inverse

- If A is $n \times n$, and there is B such that $AB = I_n$ or $BA = I_n$, then A is invertible, and B is called the **inverse** of A, denoted by A^{-1}.
- If $Ax = b$, where A is $n \times n$ invertible, then $x = A^{-1}b$ is the unique solution of the system.
- One can use the elementary row operations to change $[A|I_n]$ to $[I_n|A^{-1}]$.

Examples

Rank, linearly independence

- The **rank** of a matrix A is the number of leading ones in its row echelon form.

- Let A be a matrix. Then A and A^T have the same rank.

- The vectors $\{v_1, \dots, v_r\}$ in $\mathbb{R}^n$ is **linearly independent** if $A = [v_1|\cdots|v_r]$ has rank r so that the linear system $Ax = 0$ has only the trivial solution $[x_1, \dots, x_r]^T = [0, \dots, 0]^T$.

Examples

• The **determinant** of an $n \times n$ matrix $A = [a_{ij}]$ is defined by

$$\det(A) = a_{11} \det(A_{11}) - a_{12} \det(A_{12}) + a_{13} \det(A_{13}) - \cdots + (-1)^{n+1} \det(A_{1n}).$$

• An $n \times n$ matrix is invertible if and only if its determinant is nonzero, equivalently, the columns (rows) of A form a linearly independent set.

Examples