Math 323 Operations Research **Notes on Chapter 3.** C.K. Li

Basic format and terminology

- A linear programming (LP) problem is an optimization problem which can be formulated with a **linear objective function** and **linear inequalities and equalities** constraints on some **nonnegative** (decision) variables.
- Specifically, if the (decision) variables are $x_1, \ldots, x_n$, then the problem takes the form:

$$\text{maximize/minimzie} \quad Z = c_1x_1 + \cdots + c_nx_n$$

Subject to the constraints

$$\begin{aligned} a_{i1}x_1 + \cdots + a_{in}x_n &\leq b_i, \qquad i = 1, \ldots, p, \\ a_{i1}x_1 + \cdots + a_{in}x_n &= b_i, \qquad i = p+1, \ldots, p+q, \\ x_1, \ldots, x_n &\geq 0. \end{aligned}$$

Remarks

1. The change of objective function varies as c_ix_i for each $i = 1, \ldots, n$, and the variables are independent.
2. The values c_i's, b_j's and a_{ij}'s are all determined with certainly, i.e., not a function of time or subject to noise.
3. To solve the LP problem, we need to determine whether the **feasible region**, i.e., the region containing $(x_1, \ldots, x_n)$ satisfying the constraints, is non-empty.
4. Then find a point in the feasible region (if non-empty) that produce the **optimal solution** (if exists).
5. It is easy to change a minimization problem to a maximization problem, and vice versa
6. It is easy to change a lower bound constraint $x_i \geq \ell_i$ to a nonnegative constraint.
7. It is easy to convert one equality constraint to two inequality constraints.

3.2 Graphical solution

Example Giapetto's Woodcraving, Inc. produces

- Two toys - soldiers and trains.
- Profit equals selling price minus material cost and production cost (measure in capentry hours).
- For soldiers \$(27-10-14) = \$3; for trains \$(21-9-10) = \$2.
- Every week, no more than 100 unit of material can be used, where every soldier requires 2 unit of material, every train requires 1 unit of material.
- Every week no more than 80 carpentry hours can be used, where where every soldier requires 1 hour, every train requires 1 hour.
- Also, no more than 40 soldiers are needed every week.
- We can set up the LP program by setting x_1, x_2 as the numbers of soldiers and trains made in a week.

$$\max \ Z = 3x_1 + 2x_2$$

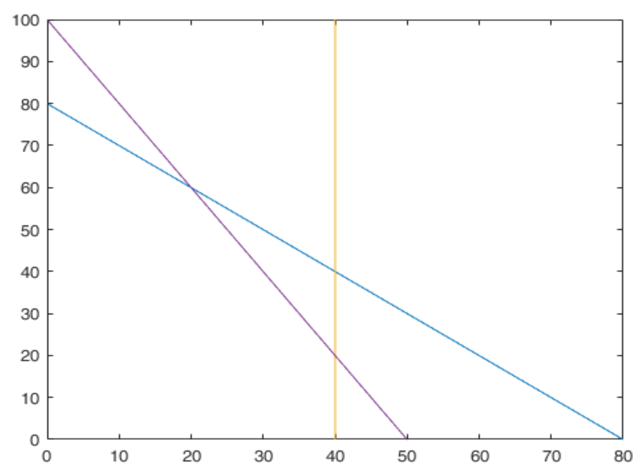
subject to

$$2x_1 + x_2 \leq 100$$

$$x_1 + x_2 \leq 80$$

$$x_1 \quad \leq 40$$

$$x_1, x_2 \geq 0.$$



1. We consider lines of the form $Z = 3x_1 + 2x_2$ and determine the maximum Z.
2. The (x_1, x_2) values on the same line will yield the same profit Z; so the line is called **isoprofit line**. In other problem, it has other name, say, **isocost line**.
3. For our problem, the optimal value Z occurs at $(x_1, x_2) = (20, 60)$.
4. At the optimal solution, there are binding constraints (equality holds) and unbinding constraints (strict inequality holds).

3.2 Some terminology associated with convex sets

1. The feasible solution set is a convex set, i.e., a line segment joint two points in the set lie in the set.

2. Examples.

3. An extreme point of a convex set is a point that is not the mid-point of two different points in the set. (Any line segment in the set containing the point must have the point as an end points.)

4. The feasible solution set is actually polygonal, i.e., defined by a finite set of linear inequalities.

5. The extreme points of a polygonal set is a vertex or corner point.

3.3 Special cases

- It may happen that there are multiple optimal solutions.

- It may happen that there is no feasible solution.

- It may happen that there are unbounded solutions.

3.4 – 3.11 Many specific (LP) modeling problems.