Math 323 Operations Research **Notes on Chapter 4.** C.K. Li

The Simplex Algorithms

§4.1 - 4.6 The basic procedures and the theory behind

Standard form

To apply simplex method, we will first change the problem to a standard form:

$$\max Z = c_1x_1 + \cdots + c_mx_n$$

Subject to

$$a_{i1}x_1 + \cdots + a_{in}x_n = b_i, \qquad i = 1, \ldots, m.$$

$$x_1, \ldots, x_n \geq 0.$$

- For

  $$\min Z = c_1x_1 + \cdots + c_nx_n$$

  we can change it to

  $$\max(-Z) = -(c_1x_1 + \cdots + c_nx_n).$$

- For the inequalities

  $$c_{i1}x_1 + \cdots + c_{in}x_n \leq b_i, \quad i = 1, \ldots, p,$$

  we can add slack variables $s_1, \ldots, s_p \geq 0$ to get

  $$c_{i1}x_1 + \cdots + c_{in}x_n + s_i = b_i, \quad i = 1, \ldots, p.$$

- For the inequalities

  $$c_{i1}x_1 + \cdots + c_{in}x_n \geq b_i, \quad i = p+1, \ldots, m,$$

  we can subtract excess variables $e_{p+1}, \ldots, e_m \geq 0$ to get

  $$c_{i1}x_1 + \cdots + c_{in}x_n - e_i = b_i, \quad i = p+1, \ldots, m.$$

Simplex algorithm when $m \leq n$

Step 1. Start with a basic feasible solution (**if it exists**).

Step 2. Improve the basic feasible solution by finding another basic solution (**if possible**) by changing one of the basic variable to go to an adjacent basic solution.

Step 3. Repeat Step 2 until we cannot improve our solution.

Example $\max Z = 5x_1 + 2x_2 + 3x_3 - x_4 + x_5$

Subject to:

$$x_1 + 2x_2 + 2x_3 + x_4 \qquad = 8$$

$$3x_1 + 4x_2 + x_3 + \quad + x_5 = 7$$

$$x_1, \ldots, x_5 \geq 0.$$

Step 1. Let $(x_4, x_5) = (8, 7)$, $x_1 = x_2 = x_3 = 0$.

Step 2. Let us bring in the nonbasic variable x_1, and consider

$$x_1 + x_4 = 8, \quad 3x_1 + x_5 = 7.$$

Increase $x_1 = 1$ to get the solution $(x_1, x_4, x_5) = (1, 7, 4)$ and $x_2 = x_3 = 0$.
The change in Z value:

$$[5(1) + 2(0) + 3(0) - 1(7) + 1(4)] - [5(0) + 2(0) + 3(0) - 1(8) + 1(7)] = 2 - (-1) = 3,$$

which is an improvement known as the relative profit of the nonbasic variable x_1.
The maximum change limited by the change of x_4, x_5:

$$x_1 + x_4 = 8 \text{ implies } x_1 \leq 8; \quad 3x_1 + x_5 = 7 \text{ implies } x_1 \leq 7/3.$$

So, we may consider the new basic solution $(x_1, x_4) = (7/3, 17/3)$ and $x_2 = x_3 = x_5 = 0$.
The value Z will increase by $(7/3) * 3 = 7$.
Now repeat Step 2 if we can improve.

Step 3. If we cannot improve for any adjacent basic feasible solution, then we get an optimal solution. (**Proof?**)

Simplex methods in Tableau form

C_B	B	$(+5)x_1$	$(+2)x_2$	$(+3)x_3$	$(-1)x_4$	$(+1)x_5$	constraints
-1	x_4	1	2	2	1	0	8
1	x_5	3	4	1	0	1	7

$$Z = (-1, 1)\begin{pmatrix}8\\7\end{pmatrix} = -1.$$

Check the relative profits for different nonbasic variables:

$$\tilde{C}_1 = 5 - (-1, 1)\begin{pmatrix}1\\3\end{pmatrix} = 3, \quad \tilde{C}_2 = 2 - (-1, 1)\begin{pmatrix}2\\4\end{pmatrix} = 0, \quad \tilde{C}_3 = 3 - (-1, 1)\begin{pmatrix}2\\1\end{pmatrix} = 4.$$

We can include the information in the tableau.

C_B	B	$(+5)x_1$	$(+2)x_2$	$(+3)x_3$	$(-1)x_4$	$(+1)x_5$	constraints
-1	x_4	1	2	2	1	0	8
1	x_5	3	4	1	0	1	7
	$\tilde{C}$	3	0	4	0	0	$Z = -1$

Because $\tilde{C}_3$ is largest, we bring in the nonbasic variable x_3.

The limit for x_3 increase is determined by the quotients of the entries in $\begin{pmatrix}x_4\\x_5\end{pmatrix} = \begin{pmatrix}8\\7\end{pmatrix}$ divided by those of the vector under x_3 yielding $\begin{pmatrix}8/2\\7/1\end{pmatrix}$. So, we can increase x_3 by 4, and change x_4 to 0.

We can now update the tableau to by Gaussian elimination:

C_B	B	$(+5)x_1$	$(+2)x_2$	$(+3)x_3$	$(-1)x_4$	$(+1)x_5$	constraints
3	x_3	1/2	1	1	1/2	0	4
1	x_5	5/2	3	0	-1/2	1	3
	$\tilde{C}$	1	-4	0	-2	0	$Z = 15$

So, we can use x_1 as a basic variable. The increase of x_1 is limited by

$$8 \text{ for } x_3, \qquad 6/5 \text{ for } x_5.$$

We can increase $x_1 = 6/5$ and decrease $x_5 = 0$.

Update the tableau to by Gaussian elimination:

C_B	B	$(+5)x_1$	$(+2)x_2$	$(+3)x_3$	$(-1)x_4$	$(+1)x_5$	constraints
3	x_3	0	2/5	1	3/5	-1/5	17/5
5	x_1	1	6/5	0	-1/5	2/5	6/5
	$\tilde{C}$	0	-26/5	0	-9/5	-2/5	$Z = 81/5$

As $\tilde{C}$ are all non-positive, we have an optimal solution.

Summary of Computational Steps

Step 1 Set up the problem in the standard form

$\max Z = c \cdot x = (c_1, \dots, c_n) \cdot (x_1, \dots, x_n) = c_1 x_1 + \cdots + c_n x_n$

Subject to $\quad Ax = b$ for an $m \times n$ matrix $A = (a_{ij})$ and $b \in \mathbb{R}^m$.

$$x_1, \dots, x_n \geq 0.$$

Step 2 Find a feasible solution using m basic variables, corresponding to the columns of A forming the invertible matrix B. In fact, if we know B, the center part of the updated tableau (without the first and last row, the first and last column) has the form

$$B^{-1}[A|b].$$

Step 3 Compute $\tilde{C}$ vector. Again, it will be of the form $\tilde{C} = c - (c_B)B^{-1}A$.

Step 4 If $\tilde{c}$ is non-positive, then we have the optimal solution.

Otherwise, go to Step 5.

Step 5 Let $\tilde{c}_i$ be the maximum value in $\tilde{c}$ corresponding to the ith nonbasic variable, and let A_i be the ith column of A.

Let $B^{-1}b = \tilde{b} = (\tilde{b}_1, \dots, \tilde{b}_m)^T$ and $B^{-1}A_i = (\tilde{a}_1, \dots, \tilde{a}_n)^T$. Then the reduced system with the potential new basic variable has augmented matrix

$$[\tilde{a}|I_m|\tilde{b}]$$

In case all $\tilde{a}_i \leq 0$, then we can increase x_i as much as possible, and we will get an **unbounded** solution.

If $\tilde{a}_j > 0$ for some $j = 1, \dots, m$, let $\tilde{b}_j / \tilde{a}_j$ be the minimum and replace the the basic variable x_j by the nonbasic variable x_i.

Return to Step 2.

Example (of unbounded solution) If in a maximization problem involving $x_1, \ldots, x_5 \geq 0$ satisfying 2 equations.

Suppose $(x_1, x_2, x_3, x_4, x_5) = (0, 0, 0, 8, 7)$ is a basic feasible solution. If $\tilde{C}_1 = 1 > 0$ and the reduced system is

$$-x_1 + x_4 = 8, \qquad -3x_1 + x_5 = 7.$$

Then we can increase x_1 indefinitely, and conclude that we have an unbounded solution.

Example of unbounded solution arising in the iteration process

C_B	B	$(2)x_1$	$(+3)x_2$	$(+0)x_3$	$(+0)x_4$	constraints
0	x_3	1	-1	1	0	2
0	x_4	-3	1	0	1	4
	$\tilde{C}$	2	3	0	0	$Z = 0$

C_B	B	$(2)x_1$	$(+3)x_2$	$(+0)x_3$	$(+0)x_4$	constraints
0	x_3	-2	0	1	1	6
3	x_2	-3	1	0	1	4
	$\tilde{C}$	11	0	0	-3	$Z = 12$

Special cases one may encounter

Alternate Optima

Suppose the iteration leads to:

C_B	B	$(+3)x_1$	$(+2)x_2$	$(+0)x_3$	$(+0)x_4$	$(+0)x_5$	constraints
0	x_3	0	0	1	-1/5	8/5	6
2	x_2	0	1	0	1/5	-3/5	1
3	x_1	1	0	0	1/5	2/5	4
	$\tilde{C}$	0	0	0	-1	0	$Z = 14$

The non-basic variable x_5 has zero relative profit. We can use it to replace x_3 and get

C_B	B	$(+3)x_1$	$(+2)x_2$	$(+0)x_3$	$(+0)x_4$	$(+0)x_5$	constraints
0	x_5	0	0	5/8	-1/8	1	15/4
2	x_2	0	1	3/8	1/8	0	13/4
3	x_1	1	0	-1/4	1/4	0	5/2
	$\tilde{C}$	0	0	0	-1	0	$Z = 14$

Unique optimum

If all the non-basic variables has negative relative profit, then the problem has a unique solution.

Ties in the selection of non-basic variable

Suppose $\tilde{c}_i$ is maximum and in the selection of basic variable x_j to be replaced, there are ties in the minimum ratio $\tilde{b}_j/\tilde{a}_j$, then we can choose any one of the x_j to be replaced.

Other degeneracy occurs when a basic variable $x_i = 0$. Then a change of basic variable may lead to no improvement even if we apply the simplex algorithm.

Example

C_B	B	$(0)x_1$	$(+0)x_2$	$(+0)x_3$	$(+2)x_4$	$(+0)x_5$	$(+3/2)x_6$	constraints
0	x_1	1	0	0	1*	-1	0	2
0	x_2	0	1	0	2	0	1	4
0	x_3	0	0	1	1	1	1	3
	$\tilde{C}$	0	0	0	2	0	3/2	$Z = 0$

C_B	B	$(0)x_1$	$(+0)x_2$	$(+0)x_3$	$(+2)x_4$	$(+0)x_5$	$(+3/2)x_6$	constraints
2	x_4	1	0	0	1	-1	0	2
0	x_2	-2	1	0	0	2*	1	0
0	x_3	-1	0	1	0	2	1	1
	$\tilde{C}$	-2	0	0	0	2	3/2	$Z = 4$

C_B	B	$(0)x_1$	$(+0)x_2$	$(+0)x_3$	$(+2)x_4$	$(+0)x_5$	$(+3/2)x_6$	constraints
2	x_4	0	1/2	0	1	0	1/2	2
0	x_5	-1	1/2	0	0	1	1/2	0
0	x_3	1	-1	1	0	0	0	1
	$\tilde{C}$	0	-1	0	0	2	1/2	$Z = 4$

- Two more iterations yield $(x_1, x_4, x_6) = (1, 1, 2)$ as the optimal solution with $Z = 5$.

- In really bad situation, we might even have cycling issue.

- In practice, we are safe if basic feasible solutions always have positive entries (non-degenerate problems).

- As long as there is a positive $\tilde{c}_i$, we should try to improve the solution though it might increase the number of steps in the calculation, but it will not affect the optimal value Z.

§4.12 Finding an initial solution - Big M method, and detecting infeasible problem

Suppose we solve an LP problem

$$\max Z = c_1x_1 + \cdots + c_nx_n$$

subject to

$$Ax = b, \qquad x_1, \ldots, x_n \geq 0,$$

where A is $m \times n$ with $m \leq n$.

If we have no obvious initial basic feasible solution, we can introduce artificial variable $a_1, \ldots, a_r \geq 0$ and study the problem

$$\max Z = c_1x_1 + \cdots + c_nx_n - M(a_1 + \cdots + a_r)$$

subject to

$$a_{i1}x_1 + \cdots + a_{in}x_n + a_i = b_i, \qquad i \in R,$$

$$a_{i1}x_1 + \cdots + a_{in}x_n = b_j, \qquad i \notin R,$$

$$x_1, \ldots, x_n, a_1, \ldots, a_r \geq 0$$

for a very large $M > 0$, and a suitable subset of $R \subseteq \{1, \ldots, m\}$.

Example $\max Z = 2x_1 + 3x_3$ subject to

$$x_1 + x_2 \leq 8, \quad x_1 + 3x_2 \geq 20, \quad x_1, x_2 \geq 0.$$

Then ...

If we cannot get rid to the artificial variable $a_1, \ldots, a_n$ at the end of the process, we have an infeasible problem!

An example of using big M method for a minimization problems

Consider $\min Z = -3x_1 + x_2 + x_3 + Mx_6 + Mx_7$

subject to

$$x_1 - 2x_2 + x_3 \leq 11, \quad -4x_1 + x_2 + 2x_3 \geq 3, \quad 2x_1 - x_3 = -1, \quad x_1, x_2, x_3 \geq 0.$$

Adding slack variable $x_4 \geq 0$, excess variable $x_5 \geq 0$ and artificial variables $x_6, x_7 \geq 0$, we get an initial basic feasible solution.

C_B	B	$(-3)x_1$	$(+1)x_2$	$(+1)x_3$	$(+0)x_4$	$(+0)x_5$	$(+M)x_6$	$(+M)x_7$	constraints
0	x_4	1	-2	1	1	0	0	0	11
M	x_6	-4	1	2	0	-1	1	0	3
M	x_7	-2	0	1*	0	0	0	1	1
	$\tilde{C}$	-3+6M	1-M	1-3M	0	M	0	0	$Z = 4M$

C_B	B	$(-3)x_1$	$(+1)x_2$	$(+1)x_3$	$(+0)x_4$	$(+0)x_5$	$(+M)x_6$	$(+M)x_7$	constraints
0	x_4	3	-2	0	1	0	0	-1	10
M	x_6	0	1*	0	0	-1	1	-2	1
1	x_3	-2	0	1	0	0	0	1	1
	$\tilde{C}$	-1	1-M	0	0	M	0	3M-1	$Z = M + 1$

C_B	B	$(-3)x_1$	$(+1)x_2$	$(+1)x_3$	$(+0)x_4$	$(+0)x_5$	$(+M)x_6$	$(+M)x_7$	constraints
0	x_4	3*	0	0	1	-2	2	-5	12
1	x_2	0	1	0	0	-1	1	-2	1
1	x_3	-2	0	1	0	0	0	1	1
	$\tilde{C}$	-1	0	0	0	1	M-1	M+1	$Z = 2$

C_B	B	$(-3)x_1$	$(+1)x_2$	$(+1)x_3$	$(+0)x_4$	$(+0)x_5$	$(+M)x_6$	$(+M)x_7$	constraints
-3	x_1	1	0	0	1/3	-2/3	2/3	-5/3	4
1	x_2	0	1	0	0	-1	1	-2	1
1	x_3	0	0	1	2/3	-4/3	4/3	-7/3	9
	$\tilde{C}$	0	0	0	1/3	1/3	M-1/3	M-2/3	$Z = -2$

§4.13 The two-phase method

Phase one

C_B	B	$(0)x_1$	$(0)x_2$	$(0)x_3$	$(+0)x_4$	$(+0)x_5$	$(+1)x_6$	$(+1)x_7$	constraints
0	x_4	1	-2	1	1	0	0	0	11
1	x_6	-4	1	2	0	-1	1	0	3
1	x_7	-2	0	1*	0	0	0	1	1
	$\tilde{C}$	6	-1	-3	0	1	0	0	$Z = 4$

C_B	B	$(0)x_1$	$(+0)x_2$	$(+0)x_3$	$(+0)x_4$	$(+0)x_5$	$(+1)x_6$	$(+1)x_7$	constraints
0	x_4	3	-2	0	1	0	0	-1	10
1	x_6	0	1*	0	0	-1	1	-2	1
0	x_3	-2	0	1	0	0	0	1	1
	$\tilde{C}$	-1	-1	0	0	1	0	3	$Z = 1$

C_B	B	$(0)x_1$	$(+0)x_2$	$(+0)x_3$	$(+0)x_4$	$(+0)x_5$	$(+1)x_6$	$(+1)x_7$	constraints
0	x_4	3	0	0	1	-2	2	-5	12
0	x_2	0	1	0	0	-1	1	-2	1
0	x_3	-2	0	1	0	0	0	1	1
	$\tilde{C}$	-1	0	0	0	1	1	1	$Z = 0$

Now move to phase two.

C_B	B	$(-3)x_1$	$(+1)x_2$	$(+1)x_3$	$(+0)x_4$	$(+0)x_5$	constraints
0	x_4	3*	0	0	1	-2	12
1	x_2	0	1	0	0	-1	1
1	x_3	-2	0	1	0	0	1
	$\tilde{C}$	-1	0	0	0	1	$Z = 2$

C_B	B	$(-3)x_1$	$(+1)x_2$	$(+1)x_3$	$(+0)x_4$	$(+0)x_5$	constraints
-3	x_1	1	0	0	1/3	-2/3	4
1	x_2	0	1	0	0	-1	1
1	x_3	0	0	1	2/3	-4/3	9
	$\tilde{C}$	0	0	0	1/3	1/3	$Z = -2$

Remarks

- We may also consider variables without sign restriction. See §4.14.
- There are other methods for solving LP: Karmarkar's method, interior point method.

  For example, see §4.15 and wikipedia.
- One can use build in Matlab commands.

  See https://www.mathworks.com/help/optim/ug/linprog.html

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To solve minimization problem
                min Z = c_1 x_1 + ... + c_n x_n
  Subject to    Ax \leq b,    AAx = bb,    L \leq x \leq U.
Input c = [c_1, ...., c_n], A, b, AA, bb, L, U.
If no inequality constraints, set A = [], b = [].
Use one of the following commands
    x = linprog(c,A,b)
    x = linprog(c,A,b,AA,bb)
    x = linprog(c,A,b,AA,bb,L,U)
   [x,fval] = linprog(___)
```

Theory behind the simplex algorithm

Theorem 1 *A point in the feasible region of an LP is an extreme point if and only if it is a basic feasible solution of the LP.*

Proof. Every point in $\mathbb{R}^m$ is uniquely determine by m linearly independent equation in $\mathbb{R}^m$. □

Theorem 2 *Suppose an LP in standard form have basic feasible solutions $v_1, \ldots, v_k$. Then every point in the feasible region has the form $v = v_0 + \sum_{j=1}^k p_j v_j$, where v_0 is the zero vector or a vector in the unbounded direction, $p_1, \ldots, p_k$ are non-negative numbers summing up to one.*

Proof. By the theory of convex analysis. □

Theorem 3 If an maximization LP has an optimal solution, then it has an optimal basic feasible solution.

Proof. If an LP has an optimal solution Z^* with objective function

$$\max Z = c \cdot x = c_1 x_1 + \cdots + c_n x_n,$$

then for any unbounded direction v_0, we have $c \cdot (Mv_0) \leq Z^*$ for any $M > 0$. So, $c \cdot v_0 = 0$. So, if $v = v_0 + \sum_{j=1}^k p_j v_j$ attains the maximum, we have

$$c.(v_0 + \sum_{j=1}^k p_j v_j) = c \cdot (\sum_{j=1}^k p_j v_j) = \sum_{j=1}^k p_j c \cdot v_j \leq \max\{c \cdot v_j : 1 \leq j \leq k\}.$$ □

Variations of Simplex Algorithm

Goal programming

Suppose the feasible region for a set of constraints is empty. One may want to set up an optimization problem to minimize the damage caused by the artificial variables.

Consider Example 10 in p. 191 in the textbook.
Let x_1 be the number of minutes ad. in football game, and x_2 be the number of minutes of TV ad.
There are three targeted groups, HIM, LIP, HIW, and the Budget constraints:

$$\begin{aligned} 7x_1 + 3x_2 &\geq 40 \quad \text{(HIM constraint)} \\ 10x_1 + 5x_2 &\geq 60 \quad \text{(LIP constraint)} \\ 5x_1 + 4x_2 &\geq 35 \quad \text{(HIW constraint)} \\ 100x_1 + 60x_2 &\leq 600 \quad \text{(Budget constraint)} \\ x_1, x_2 &\geq 0. \end{aligned}$$

We can introduce deviational variables s_i^+, s_i^- and solve the following.

$$\min Z = 200s_1^- + 200s_2^- + 50s_3^-$$

subject to:

$$\begin{aligned} 7x_1 + 3x_2 + s_1^- - s_1^+ &= 40 \quad \text{(HIM constraint)} \\ 10x_1 + 5x_2 + s_2^- - s_2^+ &= 60 \quad \text{(LIP constraint)} \\ 5x_1 + 4x_2 + s_3^- - s_3^+ &= 35 \quad \text{(HIW constraint)} \\ 100x_1 + 60x_2 \qquad\qquad &\leq 600 \quad \text{(Budget constraint)} \\ x_1, x_2, +s_1^-, s_1^+, +s_2^-, s_2^+, +s_3^-, s_3^+ &\geq 0. \end{aligned}$$

Solving the LP problem, we see that $Z = 250$ with $(x_1, x_2, s_1^+, s_3^-) = (6, 0, 2, 5)$ so that Goal 1 and Goal 2 are satisfied.

If one has to pay for the extra budget we can modify the objective function and the budget constraint to:

$$\max Z = 200s_1^- + 100s_2^- + 50s_3^- + s_4^+$$

$$100x_1 + 600x_2 + s_4^- - s_4^+ = 600.$$

The solution becomes $Z = 100/3$ $(x_1, x_2, s_1^+, s_4^+) = (13, 10, 1, 100)/3$; all three goals are met.

One may also impose penalty for the value s_i^+ and add them (with suitable weights) to the objective function.

Preemptive Goad Programming

If one does not know the exact weights of the different goals, one may consider the min $Z = P_1 s_1^- + P_2 s_2^- + P_3 s_3^-$ with $P_1 >>> P_2 >>> P_3$ and add the constraints

$$z_1 - P_1 s_1^- = 0,\ z_2 - P_2 s_2^- = 0 \text{ and } z_3 - P_3 s_3^- = 0.$$

Adding $P1\cdot$ (HIM constraint), $P2\cdot$ (LIP constraint), $P3\cdot$ (HIW constraint) to these constraints, we get

$$z_1 + 7P_1 x_1 + 3P_1 x_2 - P_1 s_1^+ = 40P_1,$$

$$z_2 + 10P_2 x_1 + 5P_2 x_2 - P s_2^+ = 60P_2,$$

$$z_3 + 5P_3 x_1 + 4P_3 x_2 - P_3 s_3^+ = 35P_3.$$

Solving the LP problem, we get $(z_1, z_2, z_3) = (0, 0, 5P_3)$ and $(x_1, x_2, s_1^+, s_3^-) = (6, 0, 2, 5)$ showing that Goal 1 and Goal 2 are met.

One may change the order of P_1, P_2, P_3 and get different solutions.

Scaling of the data

For an LP: $\max Z = c \cdot x$ with $c = (c_1, \ldots, c_n)$ and $x = (x_1, \ldots, x_n)$ s.t. $Ax = b,\ x \geq 0.$

One may replace $(c, [A|b])$ by $(\gamma c, D[A|b])$ for some suitable positive constant γ and diagonal matrix D so that the entries in γc and $D(A|b)$ are of comparable magnitudes to avoid unnecessary rounding error.

The optimal solution x will not be changed, and the optimal Z value will be changed to γZ.