Sensitivity analysis and duality

After solving an LP

$$\max Z = c \cdot x \text{ subject to } Ax = b,\ x \geq 0,$$

we want to know how would the solution change if some of the given conditions change.

1. Changing the objective function coefficient of a nonbasic variable.
2. Changing the objective function coefficient of a basic variable.
3. Changing the value in b.
4. Changing the column of a nonbasic variable.
5. Adding a new variable.
6. Adding a new constraint.

If there are two variables, we can do some analysis on the graph.

Example The toy manufacturer:

$$\max\ Z = 3x_1 + 2x_2$$

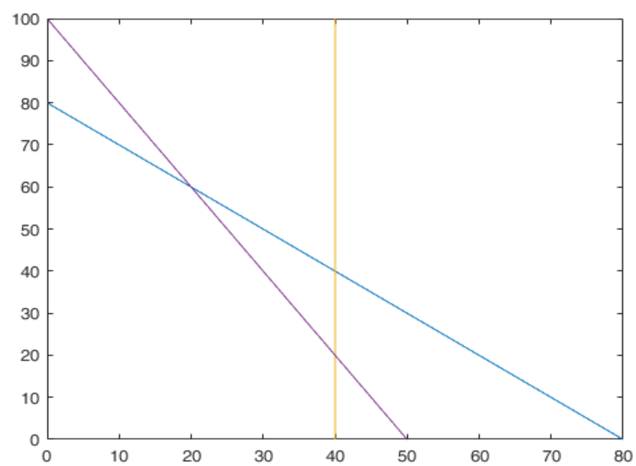
subject to

$$2x_1 + x_2 \leq 100$$

$$x_1 + x_2 \leq 80$$

$$x_1 \quad \leq 40$$

$$x_1, x_2 \geq 0.$$



Revised Simplex Method

To do the sensitivity analysis in the general setting, it is helpful to understand the revised simplex method and the dual LP/.

Set up the LP problem: $\max Z = c \cdot x$ subject to $Ax = b$ and $x \geq 0$, with an initial basic feasible solution, where $c = (c_1, \ldots, c_n), x = (x_1, \cdots, x_n)$, A is $m \times n$.

Step 1. Set B be the $m \times m$ with columns from A corresponding to the basic variables.

Step 2. Compute $\tilde{C} = c - c_B B^{-1} A$. (Only compute those $\tilde{c}_j$ corresponding to nonbasic variables.)

If $\tilde{C}$ is non-positive, we are done. Else, go to Step 3.

Step 3. Suppose A_i correspond to the maximum $\tilde{c}_i > 0$.

If $B^{-1}A_i = (\tilde{a}_1, \ldots, \tilde{a}_m)^T$ has only non-positive entries, then the problem is unbounded. Otherwise, go to Step 4.

Step 4. Let $B^{-1}b = (\tilde{b}_1, \ldots, \tilde{b}_m)^T$ and let j be such that $\tilde{b}_j/\tilde{a}_j \leq \tilde{b}_k/\tilde{a}_k$ whenever $\tilde{a}_k > 0$.
Replace the basic variable x_j by x_i. Go back to Step 2.
Here we may update the matrix B^{-1} and $B^{-1}b$ for future use:

$$B^{-1}[A_i|A_{j1} \cdots A_{j_m}|I_m|b] = [B^{-1}A_i|I_m|B^{-1}|B^{-1}b]$$

$$\rightarrow [e_j|e_1 \ldots e_{j-1}\hat{A}_j e_{j+1} \cdots e_n|\hat{B}^{-1}|\hat{B}^{-1}b].$$

Advantages of the revised simplex method

1. No need to update the whole tableau if there are many variables (columns).
2. No need to store all the information; just the original A, the basic variables, B^{-1}, and $B^{-1}b$.
3. Less error in the iteration because the original A is used in each step.
4. The method is efficient if A is sparse or having other structure.
5. The idea is useful for duality and sensitivity analysis.

Back to sensitivity analysis

Example A toy company producing three products: x_1, x_2, x_3 and set up the LP

$$\max Z = 2x_1 + 3x_2 + x_3$$

Subject to:

$$\frac{1}{3}x_1 + \frac{1}{3}x_2 + \frac{1}{3}x_3 \leq 1 \qquad \text{(labor)}$$

$$\frac{1}{3}x_1 + \frac{4}{3}x_2 + \frac{7}{3}x_3 \leq 1 \qquad \text{(material)}$$

$$x_1, x_2, x_3 \geq 0.$$

Use simplex algorithm with slack variables $x_4, x_5 \geq 0$:

C_B	B	$(+2)x_1$	$(+3)x_2$	$(+1)x_3$	$(0)x_4$	$(+1)x_5$	constraints
0	x_4	1/3	1/3	1/3	1	0	1
0	x_5	1/3	4/3	7/3	0	1	3
	$\tilde{C}$	2	3	1	0	0	$Z = 0$

$\rightarrow$

C_B	B	$(+2)x_1$	$(+3)x_2$	$(+1)x_3$	$(0)x_4$	$(+1)x_5$	constraints
2	x_1	1	0	-1	4	-1	1
3	x_2	0	1	2	-1	1	2
	$\tilde{C}$	0	0	-3	-5	-1	$Z = 8$

Case 1. Changing c_i corresponding to non-basic variables.

Note that

$$\tilde{c}_3 = c_3 - (2, 3) \cdot (-1, 2) = c_3 - 4.$$

So, if $c_3 < 4$ we have the same optimum; if $c_3 = 4$ then there may be alternate optimum; if $c_3 > 4$ then there may be improvement.

For instance, in our example, if c_3 is changed to 6, then

C_B	B	$(+2)x_1$	$(+3)x_2$	$(+6)x_3$	$(0)x_4$	$(+1)x_5$	constraints
2	x_1	1	0	-1	4	-1	1
3	x_2	0	1	2	-1	1	2
	$\tilde{C}$	0	0	2	-5	-1	$Z = 8$

$\rightarrow$

C_B	B	$(+2)x_1$	$(+3)x_2$	$(+6)x_3$	$(0)x_4$	$(+1)x_5$	constraints
2	x_1	1	1/2	0	7/2	$-1/2$	2
6	x_3	0	1/2	1	$-1/2$	1/2	1
	$\tilde{C}$	0	-1	0	-4	-2	$Z = 10$

In general, we analyze the change in $\tilde{c}_i = c_i - c_B B^{-1} A_i$.

Case 2. Changing c_j corresponding to basic variables.

If we change c_j corresponding to a basic variable x_j, then $\tilde{c} = c - c_B B^{-1} A$ in terms of c_j.

For example, if we consider a variation of c_1 after getting the basic variables x_1, x_2 for optimal in our example, then

$$(\tilde{c}_3, \tilde{c}_4, \tilde{c}_5) = (c_1 - 5, -4c_1 + 3, c_1 - 3).$$

Hence

$$\tilde{c}_3 \leq 0 \iff c_1 \leq 5 \quad \tilde{c}_4 \leq 0 \iff c_1 \geq 3/4, \quad \tilde{c}_5 \leq 0 \iff c_1 \leq 3.$$

Thus, we will use the same optimal solution $(x_1, \dots, x_5)$ if and only if $c_3 \in [3/4, 3]$. Of course, the optimal value Z will change.

The optimal solution will change otherwise. For example if $c_1 = 1$, then the optimal solution is $(x_1, x_2, x_3, x_4, x_5) = (1, 2, 0, 0, 0)$ with $Z = 7$.

If c_1 goes outside the range, we have to change $\tilde{c}$ and apply the Simplex algorithm again.

In general, we compute the entries in $\tilde{c} = c - c_B B^{-1} A$ corresponding to the non-basic variable to determine the range of change of c_j and action needed.

Case 3 Changing c in general.

If we get a solution for the basic variables x_1, x_3 and we want to change c, then we simply compute

$$\tilde{c} = c - c_B B^{-1} A = (c_1, c_2, c_3, c_4, c_5) - (c_1, c_3) \begin{pmatrix} 1 & 1/2 & 0 & 7/2 & -1/2 \\ 0 & 1/2 & 1 & -1/2 & 1/2 \end{pmatrix}$$

$$= (0, 2c_2 - c_1 - c_2, 0, 2c_4 - 7c_1 + c_3, 2c_5 + c_1 - c_3)/2.$$

and determine the course of action.

Changing the vector b

1. Note that in our example, the final tableau, the last column is $B^{-1}b$ with $B = \begin{pmatrix} 1/3 & 1/3 \\ 1/3 & 4/3 \end{pmatrix}$ so that $B^{-1} = \begin{pmatrix} 4 & -1 \\ -1 & 4 \end{pmatrix}$.

2. Now, if we change b from $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ to $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$, then the last column of the final tableau will change to
$$B^{-1}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}.$$

3. Because $B^{-1}A, \tilde{C} = c - c_B B^{-1}A$ do not change, we still have the same basic variables for the solution.

4. But (x_1, x_2, x_3) and Z change to $(5, 1, 0)$ and $Z = 13$, respectively.

5. If increasing a unit of b_1 cost \$4 (overtime cost), and the profit will increase by $\$(13-8) = \$5$. So, it is worth doing the overtime.

6. We will call the profit change corresponding to a unit change of the b_i the **shadow price**.

7. Knowing the shadow price will tell us whether it is worthwhile to increase b_i.

8. In the final tableau, the shadow price corresponding to b_i is computed by
$$c_B B^{-1}(b + e_i) - c_B B^{-1}b = c_B B^{-1}e_i.$$
Thus, the entries in the row $C_B B^{-1}$ tell us the shadow price for each of the m basic variables.

9. In our example, if we let $b^* = \begin{pmatrix} b_1 \\ 3 \end{pmatrix}$, then in the final tableau the last column becomes $B^{-1}b^* = \begin{pmatrix} 4b_1 - 3 \\ -b_1 + 3 \end{pmatrix}$.

10. So, x_1, x_2 are the basic variables for the optimal solution if $3/4 \le b_1 \le 3$.

11. The optimal value will be $Z = 2(4b_1 - 3) + 3(-b_1 + 3) = 5b_1 + 3$.

12. What if b is changed, say, to $(4, 3)^T$, so that $B^{-1}b$ is no longer feasible? Then the tableau changes to:

C_B	B	$(+2)x_1$	$(+3)x_2$	$(+1)x_3$	$(0)x_4$	$(+1)x_5$	constraints
2	x_1	1	0	-1	4	-1	13
3	x_2	0	1	2	-1^*	1	-1
	$\tilde{C}$	0	0	-3	-5	-1	$Z = 8$

We will tackle this with dual LP theory.

Changing the matrix A.

Case 1. Adding a new decision variable x_{n+1}.

In our example if we add another product, say, x_6 with a unit profit \$3, and costing 1 unit of labor and 1 unit of material. Then we update the c vector by adding $c_6 = 3$, and add the column $A_6 = (1,1)^T$ in A corresponding to x_6 and compute

$$\tilde{c}_6 = c_6 - c_B B^{-1} A_6 = 3 - (2,3)B^{-1}A_6 = 3 - (5,1) \cdot (1,1) = -3.$$

Because $\tilde{c}_6 \leq 0$, we still have the same optimal solution. If $\tilde{c}_6 < 0$, then we run the simplex algorithm.

Case 2. Changing the resources requirements.

We need to change A and B accordingly. The solution may no longer be feasible (even if we use the dual LP), and we may need to start all over again.

Case 3. Adding new constraints.

If a new constraint is added, say, $x_1 + 2x_2 + x_3 \leq 10$ is the limit of administrative hours.

Check whether the current optimal solution satisfies the constraint. If yes, it will remains to be an optimal solution.

If not, add a slack variable x_6 and consider the modified tableau:

C_B	B	$(+2)x_1$	$(+3)x_2$	$(+1)x_3$	$(+0)x_4$	$(+1)x_5$	$(+0)x_6$	constraints
2	x_1	1	0	-1	4	-1	0	1
3	x_2	0	1	2	-1	1	0	2
0	x_6	1	2	1	0	0	1	4
	$\tilde{C}$	0	0	-3	-5	-1	0	

	C_B	B	$(+2)x_1$	$(+3)x_2$	$(+1)x_3$	$(+0)x_4$	$(+1)x_5$	$(+0)x_6$	constraints
	2	x_1	1	0	-1	4	-1	0	1
$\rightarrow$	3	x_2	0	1	2	-1	1	0	2
	0	x_6	0	0	-2	-2	-1^*	1	-1
		$\tilde{C}$	0	0	-3	-5	-1	0	

where the dual LP theory can be used.

Dual LP

Consider the following primal LP:

$$\max Z = (c_1, \ldots, c_n) \cdot (x_1, \ldots, x_n) \quad \text{subject to} \quad Ax \le (b_1, \ldots, b_m)^T, \quad x_1, \ldots, x_n \ge 0,$$

where $x = (x_1, \ldots, x_n)^T$ and $A = (a_{ij})$ is $m \times n$.

Then the dual LP is defined as

$$\min W = (b_1, \ldots, b_m) \cdot (y_1, \ldots, y_m) \quad \text{subject to} \quad A^T y \ge (c_1, \ldots, c_n)^T, \quad y_1, \ldots, y_m \ge 0,$$

where $y = (y_1, \ldots, y_m)^T$.

Example (Dekota problem, p. 296) Primal problem

$$\max Z = 60x_1 + 30x_2 + 20x_3$$

$$\begin{aligned} \text{subject to:} \quad & 8x_1 + 6x_2 + x_3 \le 48 && \text{(Lumber constraint)} \\ & 4x_1 + 2x_2 + 1.5x_3 \le 20 && \text{(Finishing constraint)} \\ & 2x_1 + 1.5x_2 + 0.5x_3 \le 8 && \text{(Capentry constriant)} \\ & x_1, x_2, x_3 \ge 0. \end{aligned}$$

Dual problem.

$$\min W = 48y_1 + 20y_2 + 8y_3$$

$$\begin{aligned} \text{subject to:} \quad & 8y_1 + 4y_2 + 2y_3 \ge 60 \\ & 6y_1 + 2y_2 + 1.5y_3 \ge 30 \\ & y_1 + 1.5y_2 + 0.5y_3 \ge 20 \\ & y_1, y_2, y_3 \ge 0. \end{aligned}$$

Example (Diet problem)

$$\min W = 50y_1 + 20y_2 + 30y_3 + 80y_4$$

$$\begin{aligned} \text{subject to:} \quad & 400y_1 + 200y_2 + 150y_3 + 500y_4 \ge 500 && \text{(Calorie constraint)} \\ & 3y_1 + 2y_2 \ge 6 && \text{(Chocolate constraint)} \\ & 2y_1 + 2y_2 + 4y_3 + 4y_4 \ge 10 && \text{(Sugar constraint)} \\ & 2y_1 + 4y_2 + y_3 + 5y_4 \ge 8 && \text{(Fat constraint)} \\ & y_1, y_2, y_3, y_4 \ge 0. \end{aligned}$$

The Primal problem:

$$\max Z = 500x_1 + 6x_2 + 10x_3 + 8x_4$$

$$\begin{aligned} \text{subject to:} \quad & 400x_1 + 3x_2 + 2x_3 + 2x_4 \le 50 \\ & 200x_1 + 2x_2 + 2x_3 + 4x_4 \le 20 \\ & 150x_1 + 4x_3 + x_4 \le 30 \\ & 500x_1 + 4x_3 + 5x_4 \le 80 \\ & x_1, x_2, x_3, x_4 \ge 0. \end{aligned}$$

Remark An interpretation of the dual problem.

Finding the dual LP not in standard primal form

Example

$$\begin{aligned} &\max Z = 2x_1 + x_2 \\ \text{subject to} \quad & x_1 + x_2 = 2 \\ & 2x_1 - x_2 \geq 3 \\ & x_1 - x_2 \leq 1 \\ & x_1 \geq 0 \text{ , } x_2 \text{ urs.} \end{aligned}$$

First set $x_2 = x_2^+ - x_2^-$ with $x_2^+, x_2^- \geq 0$, and convert the problem to

$$\begin{aligned} &\max Z = 2x_1 + x_2^+ - x_2^- \\ \text{subject to} \quad & x_1 + x_2^+ - x_2^- \leq 2 \\ & -x_1 - x_2^+ + x_2^- \leq -2 \\ & -2x_1 + x_2^+ - x_2^- \leq 3 \\ & x_1 - x_2^+ + x_2^- \leq 1 \\ & x_1, x_2^+, x_2^- \geq 0. \end{aligned}$$

The dual LP becomes

$$\begin{aligned} &\min W = 2y_1' - 2y_1'' + 3y_2 + 1y_3 \\ \text{subject to} \quad & y_1' - y_1'' - 2y_2 + y_3 \geq 2 \\ & y_1' - y_1'' + y_2 - y_3 \geq 1 \\ & -y_1' + y_2'' - y_2 + y_3 \geq -1 \\ & y_1', y_2'', y_2, y_3 \geq 0. \end{aligned}$$

We set $y_1 = y_1' - y_1''$ and get

The dual LP becomes:

$$\begin{aligned} &\min W = 2y_1 + 3y_2 + y_3 \\ \text{subject to} \quad & y_1 + 2y_2 + y_3 \geq 2 \\ & y_1 + y_2 - y_3 \geq 1 \\ & -y_1 - y_2 + y_3 \geq -1 \\ & y_1 \text{ urs, } y_2, y_3 \geq 0. \end{aligned}$$

Setting $y_1 = y_1' - y_1''$ and $\hat{y}_2 = -y_2$, we get

$$\begin{aligned} &\min W = 2y_1 + 3\hat{y}_2 + y_3 \\ \text{subject to} \quad & y_1 + 2\hat{y}_2 + y_3 \geq 2 \\ & y_1 - \hat{y}_2 - y_3 = 1 \\ & y_1 \text{ urs, } \hat{y}_2 \leq 0,\ y_3 \geq 0. \end{aligned}$$

General rules for converting an LP to its dual.

Primal (Maximize) $\max Z = c^T x$	Dual (Minimize) $\min W = b^T y$
A: coefficient matrix	A^T: coefficient matrix
b: Right-hand-side vector	Cost vector
c: Price vector	Right-hand-side vector
ith constraint is an equation	The dual variable y_i has urs
ith constraint is $\leq$ type	The dual variable $y_i \geq 0$
ith constraint is $\geq$ type	The dual variable $y_i \leq 0$
x_j has urs	jth dual constraint is an equation
$x_j \geq 0$	jth dual constraint is $\geq$ type
$x_j \leq 0$	jth dual constraint is $\leq$ type

Example 1 Primal LP

$$
\begin{aligned}
& \max Z = x_1 + 4x_2 + 3x_3 \\
\text{subject to} \quad & 2x_1 + 3x_2 - 5x_3 \leq 2 \\
& 3x_1 - x_2 + 6x_3 \geq 1 \\
& x_1 + x_2 + x_3 = 4 \\
& x_1 \geq 0, \ \ x_2 \leq 0, \ \ x_3 \text{ urs.}
\end{aligned}
$$

Duel LP

$$
\begin{aligned}
& \min W = 2y_1 + y_2 + 4y_3 \\
\text{subject to} \quad & 2y_1 + 3y_2 + y_3 \geq 1 \\
& 3y_1 - y_2 + y_3 \leq 4 \\
& -5y_1 + 6y_2 + y_3 = 3 \\
& y_1 \geq 0, \ \ y_2 \leq 0, \ \ y_3 \text{ urs.}
\end{aligned}
$$

Example 2 Primal LP

$$
\begin{aligned}
& \min Z = 2x_1 + x_2 - x_3 \\
\text{subject to} \quad & x_1 + x_2 - x_3 = 1 \\
& x_1 - x_2 + x_3 \geq 2 \\
& x_2 + x_3 \leq 3 \\
& x_1 \geq 0, \ \ x_2 \leq 0, \ \ x_3 \text{ urs.}
\end{aligned}
$$

Duel LP

$$
\begin{aligned}
& \max W = y_1 + 2y_2 + 3y_3 \\
\text{subject to} \quad & y_1 + y_2 \qquad \leq 2 \\
& 3y_1 - y_2 + y_3 \geq 1 \\
& -y_1 + y_2 + y_3 = -1 \\
& y_1 \text{ urs}, \ \ y_2 \geq 0, \ \ y_3 \leq 0.
\end{aligned}
$$

Remark *The dual of the dual of an LP is the original problem.*

Theorem *Consider the standard primal and dual LP*

$$\max Z = c^T x,\ Ax \leq b,\ x \geq 0 \qquad \textit{and} \qquad \min W = b^T y,\ A^T y \geq c,\ y \leq 0$$

with $c \in \mathbb{R}^n, b \in \mathbb{R}^m, A \in \mathbb{R}^{m\times n}$. *If* $x_0 \in \mathbb{R}^n, y_0 \in \mathbb{R}^m$ *are vectors in the feasible regions so that* $Z^* = c^T x_0$ *and* $W^* = b^T y_0$ *are feasible solutions of the two problems, then* $Z^* \leq W^*$.

(1) *If* $Z^* = W^*$, *then it is the common optimal solutions for the primal and dual LP's.*

(2) *Two column vectors* $x_0 \in \mathbb{R}^n, y_0 \in \mathbb{R}^m$ *in the feasible regions will give rise to the optimal solution for the two problems if and only if*

$$(y_0^T A - c^T)x_0 + y_0^T(b - Ax_0) = 0, \qquad \textit{i.e.,} \qquad (y_0^T A - c^T)x_0 = y_0^T(b - Ax_0) = 0.$$

Proof. Let x_0 and y_0 be the vectors in the feasible regions giving rise to the Z^* and W^*. Then

$$Z^* = c^T x_0 \leq (A^T y_0)^T x_0 = y_0^T A x_0 \leq y_0^T b = b^T y_0 = W^*. \tag{1}$$

The second assertion is clear.

Finally, by (1), the equality $Z^* = W^*$ holds if and only if $Z^* = c^T x = y_0^T A x_0 = y_0^T b = W^*$, i.e., $(y_0^T A - c^T)x_0 = y_0^T(b - Ax_0) = 0$. The last assertion follows. □

Condition (2) is known as the **complementary slackness principle** for LP. It can be rephrased as follows.

At the optimal solution:

- if $b - Ax_0 \in \mathbb{C}^m$ has positive entries, i.e., the non-binding constraints, then the entries in $y_0 \in \mathbb{R}^m$ equal zero;
- if $c - A^T y_0 \in \mathbb{R}^n$ has positive entries, then the entries in $x_0 \in \mathbb{R}^n$ equal zero.

Conversely, if we get two feasible solutions $x_0 \in \mathbb{R}^n, y_0 \in \mathbb{R}^m$ satisfying the condition

$$(y_0^T A - c^T)x_0 = y_0^T(b - Ax_0) = 0,$$

then $y_0^T A x_0$ is the optimal value for the two LP's.

Example 1 Primal LP

$$\begin{aligned} &\max Z = x_1 + 4x_2 + 3x_3 \\ \text{subject to} \quad & 2x_1 + 3x_2 - 5x_3 \le 2 \\ & 3x_1 - x_2 + 6x_3 \ge 1 \\ & x_1 + x_2 + x_3 = 4 \\ & x_1 \ge 0, \ \ x_2 \le 0, \ \ x_3 \text{ urs.} \end{aligned}$$

Duel LP

$$\begin{aligned} &\min W = 2y_1 + y_2 + 4y_3 \\ \text{subject to} \quad & 2y_1 + 3y_2 + y_3 \ge 1 \\ & 3y_1 - y_2 + y_3 \le 4 \\ & -5y_1 + 6y_2 + y_3 = 3 \\ & y_1 \ge 0, \ \ y_2 \le 0, \ \ y_3 \text{ urs.} \end{aligned}$$

In this example, $x_0 = (0,0,4)^T$ and $y_0 = (0,0,3)^T$ are feasible solutions such that

$$c^T x_0 = b^T y_0 = 12$$

is the optimal for both the primal and dual problems.

Clearly, the complementary slackness conditions holds:

For $A = \begin{pmatrix} 2 & 3 & -5 \\ 3 & -1 & 6 \\ 1 & 1 & 1 \end{pmatrix}$, $b - Ax_0 = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} - 4 \begin{pmatrix} -5 \\ 6 \\ 1 \end{pmatrix} = \begin{pmatrix} 22 \\ -23 \\ 0 \end{pmatrix}$, and

$$y_0^T A - c = 3(1,1,1) - (1,4,3) = (2,-1,0).$$

Example 2 Primal LP

$$\begin{aligned} &\min Z = 2x_1 + x_2 - x_3 \\ \text{subject to} \quad & x_1 + x_2 - x_3 = 1 \\ & x_1 - x_2 + x_3 \ge 2 \\ & x_2 + x_3 \le 3 \\ & x_1 \ge 0, \ \ x_2 \le 0, \ \ x_3 \text{ urs.} \end{aligned}$$

Duel LP

$$\begin{aligned} &\max W = y_1 + 2y_2 + 3y_3 \\ \text{subject to} \quad & y_1 + y_2 \le 2 \\ & 3y_1 - y_2 + y_3 \ge 1 \\ & -y_1 + y_2 + y_3 = -1 \\ & y_1 \text{ urs}, \ \ y_2 \ge 0, \ \ y_3 \le 0. \end{aligned}$$

In this example, $x_0 = (2,0,1)^T$ and $y_0 = (1,0,0)^T$ are feasible solutions such that

$$c^T x_0 = 3 > 1 = b^T y_0.$$

The two LP's should have a finite optimal solution assuming the same value.

The dual simplex method

Theorem *Consider the standard primal and dual problem. Exactly one of the following holds.*

(a) *If both problem are feasible, then both of them have optimal solutions having the same value.*

(b) *If one problem has unbounded solution, then the other problem has no feasible solution.*

(c) *Both problem are infeasible.*

Proof. Proof of (a) is tricky. Proof of (b) is easy. If (a) and (b) do not hold, then (c) holds. □

Note If P, D stand for the primal LP and dual LP.

(1) P has finite optimal if and only if D has finite optimal.

(2) if P is unbounded then D is infeasible;

(3) if D is unbounded then P is infeasible;

(4) if P is infeasible then D is unbounded or infeasible;

(5) if D is infeasible then P is unbounded or infeasible.

Solving the primal LP to get the solution for the dual LP

Consider the primal problem in standard form

$$\max Z = c^T x \qquad \text{subject to} \qquad Ax = b, \quad x \geq 0.$$

The dual LP has the form

$$\min W = b^T y, \qquad \text{subject to} \qquad A^T y \leq c, \qquad \text{all entries of } y \text{ has urs .}$$

Note that if x_0 is an basic feasible optimal solution, then $\tilde{C} = c^T - c_B^T B^{-1} A \leq 0$. If $y^T = c_B^T B^{-1}$, then

$$c^T \geq y^T A \qquad \text{and} \qquad W = y^T b = c_B^T B^{-1} b = Z.$$

So, $Z = W$ is the optimal solution of the LP's; $y = c_B^T B^{-1}$ is an optimal solution for the duel LP.

Example Primal LP

$$
\begin{aligned}
\min Z = -32x_1 + x_2 + x_3 \\
\text{subject to} \quad x_1 - 2x_2 + x_3 + x_4 \qquad &= 11 \\
-4x_1 + x_2 + 2x_3 \qquad - x_5 &= 3 \\
-2x_1 \qquad + x_3 \qquad\qquad &= 1 \\
x_1, x_2, x_3, x_4, x_5 &\geq 0
\end{aligned}
$$

Duel LP $\max W = 11y_1 + 3y_2 + y_3$

$$
\begin{aligned}
\text{subject to} \quad y_1 - 4y_2 - 2y_3 &\leq -3 \\
-2y_1 + y_2 \qquad &\leq 1 \\
y_1 + 2y_2 + y_3 &\leq 1 \\
y_1 \qquad\qquad &\leq 0 \\
-y_2 \qquad &\leq 0 \\
y_1, y_2, y_3 \text{ urs.}
\end{aligned}
$$

We can solve the primal LP to get $(x_1, x_2, x_3) = (4, 1, 9)$ with $Z = -2$.

Then $y = c_B B^{-1} = (-3, 1, 1)B^{-1} = (-1, 1, 2)/3$ is the dual optimal solution.

Duel Simplex Method: Solving the dual LP to get the solution for the primal LP

If one solves the primal LP $\max Z = c^T x$ subject to $Ax \le b$, $x \ge 0$, and get an basic feasible optimal solution, the $y = c_B B^{-1}$ is a optimal solution for the dual LP $\min Z = b^T y$ subject to $A^T y \ge c$, entries of y have unrestricted signs.

If we run into a situation that

$$\tilde{C} = c^T - c_B^T B^{-1} A \ge 0,$$

then we have the dual feasibility vector y with $y^T = c_B^T B^T$.

Case 1. If it corresponds to a primal feasible vector x, we are done.

Case 2. If not, apply the simplex algorithm to the dual problem (in the same tableau) as follows.

Step 1. Choose $\tilde{b} = B^{-1} b$ with the most negative value (shadow price), say, $\tilde{b}_r$.

Step 2. Check whether there is $\tilde{a}_{rj}$ in $\tilde{A} = B^{-1} A$ with negative coefficients. If no, the primal problem is infeasible. If yes, select $\tilde{a}_{rj}$ such that $c_j / \tilde{a}_{rj}$ is maximum among those j with $\tilde{a}_{rj} < 0$.

Example $\min Z = x_1 + 4x_2 \quad + 3x_4$

Subject to:

$$\begin{aligned} x_1 + 2x_2 - x_3 + x_4 &\ge 3 \\ -2x_1 - x_2 + 4x_3 + x_4 &\ge 2 \\ x_1, x_2, x_3, x_4 &\ge 0. \end{aligned}$$

Use excess variables x_5, x_6 to get the standard form

$$\min Z = x_1 + 4x_2 \quad + 3x_4$$

Subject to:

$$\begin{aligned} x_1 + 2x_2 - x_3 + x_4 - x_5 \quad &= 3 \\ -2x_1 - x_2 + 4x_3 + x_4 \quad - x_6 &= 2 \\ x_1, x_2, x_3, x_4, x_5, x_6 &\ge 0. \end{aligned}$$

C_B	B	$(1)x_1$	$(+4)x_2$	$(+0)x_3$	$(+3)x_4$	$(+0)x_5$	$(+0)x_6$	constraints
0	x_5	-1^*	-2	1	-1	1	0	-3
0	x_6	2	1	-4	-1	0	1	-2
	$\tilde{C}$	1	4	0	3	0	0	

Here, we choose x_1 because of the ratio of $(-1, -2, -1)$ to $(1, 4, 3)$ equals $(-1, -2, -3)$.

C_B	B	$(1)x_1$	$(+4)x_2$	$(+0)x_3$	$(+3)x_4$	$(+0)x_5$	$(+0)x_6$	constraints
1	x_1	1	2	-1	1	-1	0	3
0	x_6	0	-3	-2^*	-3	2	1	-8
	$\tilde{C}$	0	2	1	2	1	0	

Here we choose x_3 because the ratio of $(-3, -2, -3)$ to $(2, 1, 2)$ is $(-2/3, -1/2, -2/3)$.

C_B	B	$(1)x_1$	$(+4)x_2$	$(+0)x_3$	$(+3)x_4$	$(+0)x_5$	$(+0)x_6$	constraints
1	x_1	1	$7/2$	0	$5/2$	-2	$-1/2$	7
0	x_3	0	$3/2$	1	$3/2$	-1	$-1/2$	4
	$\tilde{C}$	0	$1/2$	0	$1/2$	2	$1/2$	$Z = 7$

Back to the examples in sensitivity analysis.

Example

C_B	B	$(+2)x_1$	$(+3)x_2$	$(+1)x_3$	$(0)x_4$	$(+1)x_5$	constraints
2	x_1	1	0	-1	4	-1	13
3	x_2	0	1	2	-1^*	1	-1
	$\tilde{C}$	0	0	-3	-5	-1	

$\rightarrow$

C_B	B	$(+2)x_1$	$(+3)x_2$	$(+1)x_3$	$(0)x_4$	$(+1)x_5$	constraints
2	x_1	1	4	7	0	3	9
0	x_4	0	-1	-2	1	-1	1
	$\tilde{C}$	0	-5	-13	0	-6	$Z = 18$

Example

C_B	B	$(+2)x_1$	$(+3)x_2$	$(+1)x_3$	$(+0)x_4$	$(+1)x_5$	$(+0)x_6$	constraints
2	x_1	1	0	-1	4	-1	0	1
3	x_2	0	1	2	-1	1	0	2
0	x_5	0	0	-2	-2	-1^*	1	-1
	$\tilde{C}$	0	0	-3	-5	-1	0	

$\rightarrow$

C_B	B	$(+2)x_1$	$(+3)x_2$	$(+1)x_3$	$(+0)x_4$	$(+1)x_5$	$(+0)x_6$	constraints
2	x_1	1	0	1	6	0	-1	2
3	x_2	0	1	0	-3	0	1	1
0	x_6	0	0	2	2	1	-1	1
	$\tilde{C}$	0	0	-1	-3	0	-1	$Z = 7$