Math 323 Operations Research **Notes on Chapter 9** C.K. Li

§9.1 Integer programming

- An IP in which all variables are required to be integers is called a pure integer programming problem.
- An IP in which only some of the variables are required to be integers is called a mixed integer programming problem.
- An integer programming problem in which all the variables must equal 0 or 1 is called a 0–1 IP.
- The LP obtained by omitting all integer or 0–1 constraints on variables is called the LP relaxation of the IP.
- the LP relaxation is a less constrained, or more relaxed, version of the IP.
- This means that the feasible region for any IP must be contained in the feasible region for the corresponding LP relaxation.
- So, the optimal value for the LP relaxation is better than that of the original problem.

Example Consider the following simple IP:

$$\max Z = 21x_1 + 11x_2$$

$$\text{subjecct to} \qquad 7x_1 + 4x_2 \le 13$$

$$x_1, x_2 \ge 0; x_1, x_2 \text{ integer}$$

From Figure 1, we see that the feasible region for this problem consists of the following set of points $S = \{(0,0),(0,1),(0,2),(0,3),(1,0),(1,1)\}$.

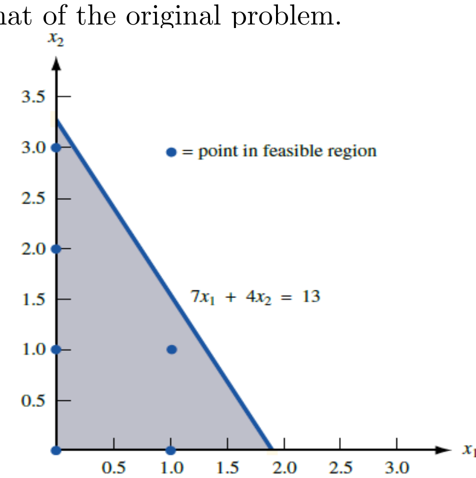


FIGURE 1
Feasible Region for Simple IP (4)

- One may solve the LP relaxation, and check all the integer points to determine the optimal.
- In our example, we find the optimal solution is $Z = 33$ with $(x_1, x_2) = (0, 3)$.
- However, it is not practical as there may be billions of integer points in the feasible regions.
- Another simple idea is to solve the LP relaxation; then round off the variables to the nearest integer for each variable.
- For our problem, the LP relaxation has optimal solution: $(x_1, x_2) = (13/7, 0)$. Rounding this solution up yields the solution $(x_1, x_2) = (2, 0)$, which is infeasible. Rounding the solution down yields the solution $(x_1, x_2) = (1, 0)$, which is not optimal.
- We need new techniques.

§9.2 Formulations

Example 1 Stockco consider 4 investments:

Investment	Net present value (NPV)	Cash outflow
1	16000	5000
2	22000	7000
3	12000	4000
4	8000	3000

Now, \$14000 is available. How to invest?

Formulation. $\max Z = 16x_1 + 22x_2 + 12x_3 + 8x_4$

subject to $5x_1 + 7x_2 + 4x_3 + 3x_4 \leq 14$.

Set $x_i = 1$ or 0 for $i = 1, 2, 3, 4$.

Exmple 2 Modify the Stockco formulation to account for each of the following requirements:

1. Stockco can invest in at most two investments.
2. If Stockco invests in investment 2, they must also invest in investment 1.
3. If Stockco invests in investment 2, they cannot invest in investment 4.

Modification

1. Simply add the constraint $x_1 + x_2 + x_3 + x_4 \leq 2$.
2. We add the constraint $x_2 \leq x_1$ or $x_2 - x_1 \leq 0$.
3. We add the constraint $x_2 + x_4 \leq 1$.

Example 3 Fixed charge IP. Gandhi Cloth Company is capable of manufacturing three types of clothing: shirts, shorts, and pants.

• The manufacture of each type of clothing requires that Gandhi have the appropriate type of machinery available.

• The machinery needed to manufacture each type of clothing must be rented at the following rates: shirt machinery, \$200 per week; shorts machinery, \$150 per week; pants machinery, \$100 per week.

• The manufacture of each type of clothing also requires the amounts of cloth and labor shown in Table 2.

• Each week, 150 hours of labor and 160 sq yd of cloth are available.

• The variable unit cost and selling price for each type of clothing are shown in Table 3.

TABLE 2
Resource Requirements for Gandhi

Clothing Type	Labor (Hours)	Cloth (Square Yards)
Shirt	3	4
Shorts	2	3
Pants	6	4

TABLE 3
Revenue and Cost Information for Gandhi

Clothing Type	Sales Price (\$)	Variable Cost (\$)
Shirt	12	6
Shorts	8	4
Pants	15	8

Formulate an IP whose solution will maximize Gandhis weekly profits.

Formulation Let x_1, x_2, x_3 be the number of shirts, shorts, and pants, produced each week;

$$y_1 = \begin{cases} 1 & \text{if any shirts are manufactured} \\ 0 & \text{otherwise} \end{cases}$$

$$y_2 = \begin{cases} 1 & \text{if any shorts are manufactured} \\ 0 & \text{otherwise} \end{cases}$$

$$y_3 = \begin{cases} 1 & \text{if any pants are manufactured} \\ 0 & \text{otherwise} \end{cases}$$

In short, if $x_j > 0$, then $y_j = 1$, and if $x_j = 0$, then $y_j = 0$. Thus, Gandhi's weekly profits = (weekly sales revenue) − (weekly variable costs) − (weekly costs of renting machinery).

We can then formulate the problem as:

$$\begin{aligned}
\max z &= 6x_1 + 4x_2 + 7x_3 - 200y_1 - 150y_2 - 100y_3 \\
\text{s.t.} \quad & 3x_1 + 2x_2 + 6x_3 \leq 150 \\
& 4x_1 + 3x_2 + 4x_3 \leq 160 \\
& x_1 \leq M_1 y_1 \\
& x_2 \leq M_2 y_2 \\
& x_3 \leq M_3 y_3 \\
& x_1, x_2, x_3 \geq 0;\ x_1, x_2, x_3 \text{ integer} \\
& y_1, y_2, y_3 = 0 \text{ or } 1
\end{aligned}$$

The optimal solution to the Gandhi problem is $z = \$75$, $x_3 = 25, y_3 = 1$. Thus, Gandhi should produce 25 pants each week.

Example 4 The Lockbox problem J. C. Nickles receives credit card payments from four regions of the country (West, Midwest, East, and South).

The average daily value of payments mailed by customers from each region is as follows:

the West, \$70,000; the Midwest, \$50,000; the East, \$60,000; the South, \$40,000.

Nickles must decide where customers should mail their payments.

Because Nickles can earn 20% annual interest by investing these revenues, it would like to receive payments as quickly as possible.

Nickles is considering setting up operations to process payments (often referred to as lockboxes) in four different cities:

Los Angeles, Chicago, New York, and Atlanta.

The average number of days (from time payment is sent) until a check clears and Nickles can deposit the money depends on the city to which the payment is mailed, as shown in Table 4.

TABLE 4

Average Number of Days from Mailing of Payment Until Payment Clears

	To			
From	City 1 (Los Angeles)	City 2 (Chicago)	City 3 (New York)	City 4 (Atlanta)
Region 1 West	2	6	8	8
Region 2 Midwest	6	2	5	5
Region 3 East	8	5	2	5
Region 4 South	8	5	5	2

For example, if a check is mailed from the West to Atlanta, it would take an average of 8 days before Nickles could earn interest on the check.

The annual cost of running a lockbox in any city is \$50,000.

Formulate an IP that Nickles can use to minimize the sum of costs due to lost interest and lockbox operations.

Let $x_{ij} \in \{0, 1\}$ so that 1 means regions i send checks to region j, and let $y_j \in \{0, 1\}$ so that 1 means that mailbox is operated at city j.

$$\min z = 28x_{11} + 84x_{12} + 112x_{13} + 112x_{14} + 60x_{21} + 20x_{22} + 50x_{23} + 50x_{24}$$
$$+ 96x_{31} + 60x_{32} + 24x_{33} + 60x_{34} + 64x_{41} + 40x_{42} + 40x_{43} + 16x_{44}$$
$$+ 50y_1 + 50y_2 + 50y_3 + 50y_4$$

s.t. $x_{11} + x_{12} + x_{13} + x_{14} = 1$ (West region constraint)

$x_{21} + x_{22} + x_{23} + x_{24} = 1$ (Midwest region constraint)

$x_{31} + x_{32} + x_{33} + x_{34} = 1$ (East region constraint)

$x_{41} + x_{42} + x_{43} + x_{44} = 1$ (South region constraint)

$x_{11} \le y_1, x_{21} \le y_1, x_{31} \le y_1, x_{41} \le y_1, x_{12} \le y_2, x_{22} \le y_2, x_{32} \le y_2, x_{42} \le y_2,$

$x_{13} \le y_3, x_{23} \le y_3, x_{33} \le y_3, x_{43} \le y_3, x_{14} \le y_4, x_{24} \le y_4, x_{34} \le y_4, x_{44} \le y_4$

All x_{ij} and $y_j = 0$ or 1

The optimal solution is $z = 242, y_1 = 1, y_3 = 1, x_{11} = 1, x_{23} = 1, x_{33} = 1, x_{43} = 1$. Thus, Nickles should have a lockbox operation in Los Angeles and New York. West customers should send payments to Los Angeles, and all other customers should send payments to New York.

There is an alternative way of modeling the Type 2 constraints. Instead of the 16 constraints of the form $x_{ij} \le y_j$, we may include the following four constraints:

$x_{11} + x_{21} + x_{31} + x_{41} \le 4y_1$ (Los Angeles constraint)

$x_{12} + x_{22} + x_{32} + x_{42} \le 4y_2$ (Chicago constraint)

$x_{13} + x_{23} + x_{33} + x_{43} \le 4y_3$ (New York constraint)

$x_{14} + x_{24} + x_{34} + x_{44} \le 4y_4$ (Atlanta constraint)

TABLE 5

Calculation of Annual Lost Interest

Assignment	Annual Lost Interest Cost (\$)
West to L.A.	0.20(70,000)2 = 28,000
West to Chicago	0.20(70,000)6 = 84,000
West to N.Y.	0.20(70,000)8 = 112,000
West to Atlanta	0.20(70,000)8 = 112,000
Midwest to L.A.	0.20(50,000)6 = 60,000
Midwest to Chicago	0.20(50,000)2 = 20,000
Midwest to N.Y.	0.20(50,000)5 = 50,000
Midwest to Atlanta	0.20(50,000)5 = 50,000
East to L.A.	0.20(60,000)8 = 96,000
East to Chicago	0.20(60,000)5 = 60,000
East to N.Y.	0.20(60,000)2 = 24,000
East to Atlanta	0.20(60,000)5 = 60,000
South to L.A.	0.20(40,000)8 = 64,000
South to Chicago	0.20(40,000)5 = 40,000
South to N.Y	0.20(40,000)5 = 40,000
South to Atlanta	0.20(40,000)2 = 16,000

Example 5 set covering problem There are six cities (cities 16) in Kilroy County. The county must determine where to build fire stations. The county wants to build the minimum number of fire stations needed to ensure that at least one fire station is within 15 minutes (driving time) of each city.

The times (in minutes) required to drive between the cities in Kilroy County are shown in Table 6. Formulate an IP that will tell Kilroy how many fire stations should be built and where they should be located.

TABLE 6
Time Required to Travel between Cities in Kilroy County

	To					
From	City 1	City 2	City 3	City 4	City 5	City 6
City 1	0	10	20	30	30	20
City 2	10	0	25	35	20	10
City 3	20	25	0	15	30	20
City 4	30	35	15	0	15	25
City 5	30	20	30	15	0	14
City 6	20	10	20	25	14	0

Formulation Let $x_i \in \{0,1\}$ so that $x_i = 1$ means that a fire station is built in city i.

TABLE 7
Cities within 15 Minutes of Given City

City	Within 15 Minutes
1	1, 2
2	1, 2, 6
3	3, 4
4	3, 4, 5
5	4, 5, 6
6	2, 5, 6

$$\min z = x_1 + x_2 + x_3 + x_4 + x_5 + x_6$$

$$
\begin{aligned}
\text{s.t.} \quad & x_1 + x_2 \geq 1 && \text{(City 1 constraint)} \\
& x_1 + x_2 + x_6 \geq 1 && \text{(City 2 constraint)} \\
& x_3 + x_4 \geq 1 && \text{(City 3 constraint)} \\
& x_3 + x_4 + x_5 \geq 1 && \text{(City 4 constraint)} \\
& x_4 + x_5 + x_6 \geq 1 && \text{(City 5 constraint)} \\
& x_2 + x_5 + x_6 \geq 1 && \text{(City 6 constraint)} \\
& x_i = 0 \text{ or } 1 \quad (i = 1, 2, 3, 4, 5, 6)
\end{aligned}
$$

One optimal solution to this IP is $z = 2$, $x_2 = x_4 = 1$, $x_1 = x_3 = x_5 = x_6 = 0$. Thus, Kilroy County can build two fire stations: one in city 2 and one in city 4.

Either–Or Constraints

The following situation commonly occurs in mathematical programming problems. We are given two constraints of the form

$$f(x_1, x_2, \ldots, x_n) \leq 0 \tag{26}$$

$$g(x_1, x_2, \ldots, x_n) \leq 0 \tag{27}$$

We want to ensure that at least one of (26) and (27) is satisfied, often called **either–or constraints.** Adding the two constraints (26′) and (27′) to the formulation will ensure that at least one of (26) and (27) is satisfied:

$$f(x_1, x_2, \ldots, x_n) \leq My \tag{26′}$$

$$g(x_1, x_2, \ldots, x_n) \leq M(1 - y) \tag{27′}$$

Example 6 Either Or Constraints Dorian Auto is considering manufacturing three types of autos: compact, midsize, and large.

The resources required for, and the profits yielded by, each type of car are shown in Table 8.

Currently, 6,000 tons of steel and 60,000 hours of labor are available.

For production of a type of car to be economically feasible, at least 1,000 cars of that type must be produced.

Formulate an IP to maximize Dorians profit.

Let x_1, x_2, x_3 be the number of compact, midsize, large cars produced.

We know that if any cars of a given type are produced, then at least 1,000 cars of that type must be produced. Thus, for $i = 1, 2, 3$, we must have $x_i \le 0$ or $x_i \ge 1{,}000$. Steel and labor are limited, so Dorian must satisfy the following five constraints:

Constraint 1 $x_1 \le 0$ or $x_1 \ge 1{,}000$.

Constraint 2 $x_2 \le 0$ or $x_2 \ge 1{,}000$.

Constraint 3 $x_3 \le 0$ or $x_3 \ge 1{,}000$.

Constraint 4 The cars produced can use at most 6,000 tons of steel.

Constraint 5 The cars produced can use at most 60,000 hours of labor.

We may replace Constraint 1 by the following:

$$
\begin{aligned}
x_1 &\le M_1 y_1 \\
1{,}000 - x_1 &\le M_1(1 - y_1) \\
y_1 &= 0 \text{ or } 1
\end{aligned}
$$

TABLE 8

Resources and Profits for Three Types of Cars

	Car Type		
Resource	**Compact**	**Midsize**	**Large**
Steel required	1.5 tons	3 tons	5 tons
Labor required	30 hours	25 hours	40 hours
Profit yielded ($)	2,000	3,000	4,000

To ensure that both x_1 and $1{,}000 - x_1$ will never exceed M_1, it suffices to choose M_1 large enough so that M_1 exceeds 1,000 and x_1 is always less than M_1. Building $\frac{60{,}000}{30} = 2{,}000$ compacts would use all available labor (and still leave some steel), so at most 2,000 compacts can be built. Thus, we may choose $M_1 = 2{,}000$.

We can apply similar argument to the second and third constraints and get the following.

$$
\begin{aligned}
\max z = 2x_1 &+ 3x_2 + 4x_3 \\
\text{s.t.} \quad x_1 &\le 2{,}000y_1 \\
1{,}000 - x_1 &\le 2{,}000(1 - y_1) \\
x_2 &\le 2{,}000y_2 \\
1{,}000 - x_2 &\le 2{,}000(1 - y_2) \\
x_3 &\le 1{,}200y_3 \\
1{,}000 - x_3 &\le 1{,}200(1 - y_3) \\
1.5x_1 + 3x_2 + 5x_3 &\le 6{,}000 \qquad \text{(Steel constraint)} \\
30x_1 + 25x_2 + 40x_3 &\le 60{,}000 \qquad \text{(Labor constraint)} \\
x_1, x_2, x_3 &\ge 0;\ x_1, x_2, x_3 \text{ integer} \\
y_1, y_2, y_3 &= 0 \text{ or } 1
\end{aligned}
$$

The optimal solution to the IP is $z = 6{,}000$, $x_2 = 2{,}000$, $y_2 = 1$, $y_1 = y_3 = x_1 = x_3 = 0$. Thus, Dorian should produce 2,000 midsize cars. If Dorian had not been required to manufacture at least 1,000 cars of each type, then the optimal solution would have been to produce 570 compacts and 1,715 midsize cars.

If–Then Constraints

In many applications, the following situation occurs: We want to ensure that if a constraint $f(x_1, x_2, \ldots, x_n) > 0$ is satisfied, then the constraint $g(x_1, x_2, \ldots, x_n) \geq 0$ must be satisfied, while if $f(x_1, x_2, \ldots, x_n) > 0$ is not satisfied, then $g(x_1, x_2, \ldots, x_n) \geq 0$ may or may not be satisfied. In short, we want to ensure that $f(x_1, x_2, \ldots, x_n) > 0$ implies $g(x_1, x_2, \ldots, x_n) \geq 0$.

To ensure this, we include the following constraints in the formulation:

$$-g(x_1, x_2, \ldots, x_n) \leq My \tag{28}$$

$$f(x_1, x_2, \ldots, x_n) \leq M(1 - y) \tag{29}$$

$$y = 0 \text{ or } 1$$

As usual, M is a large positive number. (M must be chosen large enough so that $f \leq M$ and $-g \leq M$ hold for all values of $x_1, x_2, \ldots, x_n$ that satisfy the other constraints in the problem.)

Example

To illustrate the use of this idea, suppose we add the following constraint to the Nickles lockbox problem: If customers in region 1 send their payments to city 1, then no other customers may send their payments to city 1. Mathematically, this restriction may be expressed by

$$\text{If } x_{11} = 1, \qquad \text{then} \qquad x_{21} = x_{31} = x_{41} = 0 \tag{30}$$

Because all x_{ij} must equal 0 or 1, (30) may be written as

$$\text{If } x_{11} > 0, \qquad \text{then} \qquad x_{21} + x_{31} + x_{41} \leq 0, \qquad \text{or} \qquad -x_{21} - x_{31} - x_{41} \geq 0 \tag{30$'$}$$

If we define $f = x_{11}$ and $g = -x_{21} - x_{31} - x_{41}$, we can use (28) and (29) to express (30′) [and therefore (30)] by the following two constraints:

$$x_{21} + x_{31} + x_{41} \leq My$$

$$x_{11} \leq M(1 - y)$$

$$y = 0 \text{ or } 1$$

Because $-g$ and f can never exceed 3, we can choose $M = 3$ and add the following constraints to the original lockbox formulation:

$$x_{21} + x_{31} + x_{41} \leq 3y$$

$$x_{11} \leq 3(1 - y)$$

$$y = 0 \text{ or } 1$$

LP involving piecewise linear functions

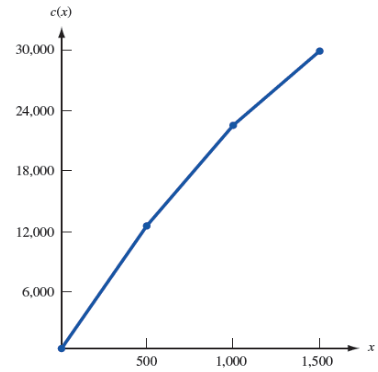


FIGURE 3
Cost of Purchasing Oil

An oil cost example. Suppose the first 500 gallons of oil purchased cost .25 per gallon; the next 500 gallons cost .20 per gallon; and the next 500 gallons cost .15 per gallon. At most, 1,500 gallons of oil can be purchased. Let x be the number of gallons of oil purchased. Then $0 \le x \le 1500$, and the cost function is:

$$f(x) = 25x, \text{ or } f(x) = 25(500) + 20(x - 500) \text{ or } f(x) = 25(500) + 20(500) + 15(x - 1000).$$

Note that the price $f(x)$ is a piecewise linear function with breaking points $b_1, \ldots, b_4$ equal to: $0, 200, 1000, 1500$.

In general, if $b_k \le x \le b_{k+1}$, then $x = z_k x b_k + (1 - z_k) b_{k+1}$ and $f(x) = z_k f(b_k) + (1 - z_k) f(b_{k+1})$.

In our example, if $x = 800$, then $500 \le x \le 100$ so that $x = 0.4(500) + 0.6(1000)$ and

$$f(x) = f(800) = 0.4f(500) + 0.6f(1000) = 18500.$$

We are now ready to describe the method used to express a piecewise linear function via linear constraints and 0–1 variables:

Step 1 Wherever $f(x)$ occurs in the optimization problem, replace $f(x)$ by

$$z_1 f(b_1) + z_2 f(b_2) + \cdots + z_n f(b_n).$$

Step 2 Add the following constraints to the problem:

$$z_1 \le y_1, z_2 \le y_1 + y_2, z_3 \le y_2 + y_3, \ldots, z_{n-1} \le y_{n-2} + y_{n-1}, z_n \le y_{n-1}$$
$$y_1 + y_2 + \cdots + y_{n-1} = 1$$
$$z_1 + z_2 + \cdots + z_n = 1$$
$$x = z_1 b_1 + z_2 b_2 + \cdots + z_n b_n$$
$$y_i = 0 \text{ or } 1 \qquad (i = 1, 2, \ldots, n - 1); \qquad z_i \ge 0 \qquad (i = 1, 2, \ldots, n)$$

We will study Example 7 and Example 8.

Example 7 Euing Gas produces two types of gasoline (gas 1 and gas 2) from two types of oil (oil 1 and oil 2).

- Each gallon of gas 1 must contain at least 50 percent oil 1, and each gallon of gas 2 must contain at least 60 percent oil 1.
- Each gallon of gas 1 can be sold for \$.12, and each gallon of gas 2 can be sold for \$.14.
- Currently, 500 gallons of oil 1 and 1,000 gallons of oil 2 are available.
- As many as 1,500 more gallons of oil 1 can be purchased at the following prices:

  first 500 gallons, \$.25 per gallon; next 500 gallons, \$.20 per gallon; next 500 gallons, \$.15 per gallon.

Formulate of an IP that will maximize Euings profits (revenues − purchasing costs):

Except for the fact that the cost of purchasing additional oil 1 is a piecewise linear function, this is a straightforward blending problem. With this in mind, we define

$$x = \text{amount of oil 1 purchased}$$
$$x_{ij} = \text{amount of oil } i \text{ used to produce gas } j \qquad (i, j = 1, 2)$$

Then (in cents)

$$\text{Total revenue} - \text{cost of purchasing oil 1} = 12(x_{11} + x_{21}) + 14(x_{12} + x_{22}) - c(x)$$

As we have seen previously,

$$c(x) = \begin{cases} 25x & (0 \le x \le 500) \\ 20x + 2{,}500 & (500 \le x \le 1{,}000) \\ 15x + 7{,}500 & (1{,}000 \le x \le 1{,}500) \end{cases}$$

Thus, Euing's objective function is to maximize

$$z = 12x_{11} + 12x_{21} + 14x_{12} + 14x_{22} - c(x)$$

Euing faces the following constraints:

Constraint 1 Euing can use at most $x + 500$ gallons of oil 1.

Constraint 2 Euing can use at most 1,000 gallons of oil 2.

Constraint 3 The oil mixed to make gas 1 must be at least 50% oil 1.

Constraint 4 The oil mixed to make gas 2 must be at least 60% oil 1.

Constraint 1 yields

$$x_{11} + x_{12} \le x + 500$$

Constraint 2 yields

$$x_{21} + x_{22} \le 1{,}000$$

Constraint 3 yields

$$\frac{x_{11}}{x_{11} + x_{21}} \ge 0.5 \quad \text{or} \quad 0.5x_{11} - 0.5x_{21} \ge 0$$

Constraint 4 yields

$$\frac{x_{12}}{x_{12} + x_{22}} \ge 0.6 \quad \text{or} \quad 0.4x_{12} - 0.6x_{22} \ge 0$$

Also all variables must be nonnegative. Thus, Euing Gas must solve the following optimization problem:

$$
\begin{aligned}
\max z = {} & 12x_{11} + 12x_{21} + 14x_{12} + 14x_{22} - c(x) \\
\text{s.t.} \quad & x_{11} + x_{12} \le x + 500 \\
& x_{21} + x_{22} \le 1{,}000 \\
& 0.5x_{11} - 0.5x_{21} \ge 0 \\
& 0.4x_{12} - 0.6x_{22} \ge 0 \\
& x_{ij} \ge 0,\ 0 \le x \le 1{,}500
\end{aligned}
$$

Because $c(x)$ is a piecewise linear function, the objective function is not a linear function of x, and this optimization is not an LP. By using the method described earlier, however, we can transform this problem into an IP. After recalling that the break points for $c(x)$ are 0, 500, 1,000, and 1,500, we proceed as follows:

Step 1 Replace $c(x)$ by $c(x) = z_1c(0) + z_2c(500) + z_3c(1{,}000) + z_4c(1{,}500)$.

Step 2 Add the following constraints:

$$
\begin{aligned}
x &= 0z_1 + 500z_2 + 1{,}000z_3 + 1{,}500z_4 \\
z_1 &\le y_1,\ z_2 \le y_1 + y_2,\ z_3 \le y_2 + y_3,\ z_4 \le y_3 \\
z_1 &+ z_2 + z_3 + z_4 = 1, \quad y_1 + y_2 + y_3 = 1 \\
y_i &= 0 \text{ or } 1\ (i = 1, 2, 3);\ z_i \ge 0\ (i = 1, 2, 3, 4)
\end{aligned}
$$

Our new formulation is the following IP:

$$
\begin{aligned}
\max z = {} & 12x_{11} + 12x_{21} + 14x_{12} + 14x_{22} - z_1c(0) - z_2c(500) \\
& - z_3c(1{,}000) - z_4c(1{,}500) \\
\text{s.t.} \quad & x_{11} + x_{12} \le x + 500 \\
& x_{21} + x_{22} \le 1{,}000 \\
& 0.5x_{11} - 0.5x_{21} \ge 0 \\
& 0.4x_{12} - 0.6x_{22} \ge 0 \\
& x = 0z_1 + 500z_2 + 1{,}000z_3 + 1{,}500z_4 && (31) \\
& z_1 \le y_1 && (32) \\
& z_2 \le y_1 + y_2 && (33) \\
& z_3 \le y_2 + y_3 && (34) \\
& z_4 \le y_3 && (35) \\
& y_1 + y_2 + y_3 = 1 && (36) \\
& z_1 + z_2 + z_3 + z_4 = 1 && (37) \\
& y_i = 0 \text{ or } 1 \qquad (i = 1, 2, 3);\ z_i \ge 0 \qquad (i = 1, 2, 3, 4) \\
& x_{ij} \ge 0
\end{aligned}
$$

- To see why this formulation works, observe that because $y_1 + y_2 + y_3 = 1, y_i \in \{0, 1\}$ and exactly one of the y_is will equal 1, and the others will equal 0.
- Now, (32)(37) imply that if $y_i = 1$, then $z_i, z_{i+1} \geq 0$, but all the other z_ℓs must equal 0.
- For instance, if $y_2 = 1$ then $y_1 = y_3 = 0$. So, (32)(35) become $z_1 \leq 0, z_2 \leq 1, z_3 \leq 1,\ z_4 \leq 0$.
- These constraints force $z_1 = z_4 = 0$ and allow $z_2, z_3 \in [0, 1]$.
- We can now show that (31)(37) correctly represent the piecewise linear function $c(x)$.
- Choose any value of x, say $x = 800$.
- Note that $b_2 = 500 \leq 800 \leq 1,000 = b_3$.
- For $x = 800$, what values do our constraints assign to y_1, y_2, and y_3?
- The value $y_1 = 1$ is impossible; otherwise, $y_2 = y_3 = 0$. Then (34)(35) force $z_3 = z_4 = 0$.
- Then (31) reduces to $800 = x \leq 500z_2$, which cannot be satisfied by $z_2 \leq 1$.
- Similarly, $y_3 = 1$ is impossible.
- If we try $y_2 = 1$ (32) and (35) force $z_1 = z_4 = 0$, then (33) and (34) imply $z_2, z_3 \in [0, 1]$.
- Now (31) becomes $800 = x \leq 500z_2 + 1,000z_3$.
- Because $z_2 + z_3 = 1$, we obtain $z_2 = 0.4$ and $z_3 = 0.6$.
- Now the objective function reduces to

$$12x_{11} + 12x_{21} + 14x_{21} + 14x_{22} - \frac{2c(500)}{5} - \frac{3c(1{,}000)}{5}$$

Because

$$c(800) = \frac{2c(500)}{5} + \frac{3c(1{,}000)}{5}$$

our objective function yields the correct value of Euing's profits!

The optimal solution to Euing's problem is $z = 12{,}500$, $x = 1{,}000$, $x_{12} = 1{,}500$, $x_{22} = 1{,}000$, $y_3 = z_3 = 1$. Thus, Euing should purchase 1,000 gallons of oil 1 and produce 2,500 gallons of gas 2.

Remarks

In general, constraints of the form (31)–(37) ensure that if $b_i \leq x \leq b_{i+1}$, then $y_i = 1$ and only z_i and z_{i+1} can be positive. Because $c(x)$ is linear for $b_i \leq x \leq b_{i+1}$, the objective function will assign the correct value to $c(x)$.

If a piecewise linear function $f(x)$ involved in a formulation has the property that the slope of $f(x)$ becomes less favorable to the decision maker as x increases, then the tedious IP formulation we have just described is unnecessary.

Example 8 Media Selection

- Dorian Auto has a $20,000 advertising budget.
- Dorian can purchase full-page ads in two magazines: Inside Jocks (IJ) and Family Square (FS).
- An exposure occurs when a person reads a Dorian Auto ad for the first time.
- The number of exposures generated by each ad in IJ is as follows:

  ads 16, 10,000 exposures; ads 710, 3,000 exposures; ads 1115, 2,500 exposures; ads 16+, 0 exposures.
- For example, 8 ads in IJ would generate $6(10,000) + 2(3,000) = 66,000$ exposures.
- The number of exposures generated by each ad in FS is as follows:

  ads 14, 8,000 exposures; ads 512, 6,000 exposures; ads 1315, 2,000 exposures; ads 16+, 0 exposures.
- Thus, 13 ads in FS would generate $4(8,000) + 8(6,000) + 1(2,000) = 82,000$ exposures.
- Each full-page ad in either magazine costs $1,000.
- Assume there is no overlap in the readership of the two magazines.

Formulation of an IP to maximize the number of exposures that Dorian can obtain with limited advertising funds.

If we define

$$
\begin{aligned}
x_1 &= \text{number of IJ ads yielding 10,000 exposures} \\
x_2 &= \text{number of IJ ads yielding 3,000 exposures} \\
x_3 &= \text{number of IJ ads yielding 2,500 exposures} \\
y_1 &= \text{number of FS ads yielding 8,000 exposures} \\
y_2 &= \text{number of FS ads yielding 6,000 exposures} \\
y_3 &= \text{number of FS ads yielding 2,000 exposures}
\end{aligned}
$$

then the total number of exposures (in thousands) is given by

$$10x_1 + 3x_2 + 2.5x_3 + 8y_1 + 6y_2 + 2y_3$$

Thus, Dorian wants to maximize

$$z = 10x_1 + 3x_2 + 2.5x_3 + 8y_1 + 6y_2 + 2y_3$$

Because the total amount spent (in thousands) is just the toal number of ads placed in both magazines, Dorian's budget constraint may be written as

$$x_1 + x_2 + x_3 + y_1 + y_2 + y_3 \leq 20$$

The statement of the problem implies that $x_1 \leq 6$, $x_2 \leq 4$, $x_3 \leq 5$, $y_1 \leq 4$, $y_2 \leq 8$, and $y_3 \leq 3$ all must hold. Adding the sign restrictions on each variable and noting that each variable must be an integer, we obtain the following IP:

$$
\begin{aligned}
\max z = {} & 10x_1 + 3x_2 + 2.5x_3 + 8y_1 + 6y_2 + 2y_3 \\
\text{s.t.} \quad & x_1 + x_2 + x_3 + y_1 + y_2 + y_3 \leq 20 \\
& x_1 \leq 6 \\
& x_2 \leq 4 \\
& x_3 \leq 5 \\
& y_1 \leq 4 \\
& y_2 \leq 8 \\
& y_3 \leq 3 \\
& x_i, y_i \text{ integer} \quad (i = 1, 2, 3) \\
& x_i, y_i \geq 0 \quad (i = 1, 2, 3)
\end{aligned}
$$

Observe that the statement of the problem implies that x_2 cannot be positive unless x_1 assumes its maximum value of 6. Similarly, x_3 cannot be positive unless x_2 assumes its maximum value of 4. Because x_1 ads generate more exposures than x_2 ads, however, the act of maximizing ensures that x_2 will be positive only if x_1 has been made as large as possible. Similarly, because x_3 ads generate fewer exposures than x_2 ads, x_3 will be positive only if x_2 assumes its maximum value. (Also, y_2 will be positive only if $y_1 = 4$, and y_3 will be positive only if $y_2 = 8$.)

The optimal solution to Dorian's IP is $z = 146{,}000$, $x_1 = 6$, $x_2 = 2$, $y_1 = 4$, $y_2 = 8$, $x_3 = 0$, $y_3 = 0$. Thus, Dorian will place $x_1 + x_2 = 8$ ads in IJ and $y_1 + y_2 = 12$ ads in FS.

In Example 8, additional advertising in a magazine yielded diminishing returns. This ensured that x_i (y_i) would be positive only if x_{i-1} (y_{i-1}) assumed its maximum value. If additional advertising generated increasing returns, then this formulation would not yield the correct solution. For example, suppose that the number of exposures generated by each IJ ad was as follows: ads 1–6, 2,500 exposures; ads 7–10, 3,000 exposures; ads 11–15, 10,000 exposures. Suppose also that the number of exposures generated by each FS is as follows: ads 1–4, 2,000 exposures; ads 5–12, 6,000 exposures; ads 13–15, 8,000 exposures.

If we define

$$
\begin{aligned}
x_1 &= \text{number of IJ ads generating 2,500 exposures}\\
x_2 &= \text{number of IJ ads generating 3,000 exposures}\\
x_3 &= \text{number of IJ ads generating 10,000 exposures}\\
y_1 &= \text{number of FS ads generating 2,000 exposures}\\
y_2 &= \text{number of FS ads generating 6,000 exposures}\\
y_3 &= \text{number of FS ads generating 8,000 exposures}
\end{aligned}
$$

the reasoning used in the previous example would lead to the following formulation:

$$
\begin{array}{ll}
\max z = 2.5x_1 + 3x_2 + 10x_3 + 2y_1 + 6y_2 + 8y_3 & \\
\text{s.t.} \quad x_1 + x_2 + x_3 + y_1 + y_2 + y_3 \le 20 & \\
\qquad\quad x_1 \le 6 & \\
\qquad\quad x_2 \le 4 & \\
\qquad\quad x_3 \le 5 & \\
\qquad\quad y_1 \le 4 & \\
\qquad\quad y_2 \le 8 & \\
\qquad\quad y_3 \le 3 & \\
\qquad\quad x_i, y_i \text{ integer} \quad (i = 1, 2, 3) & \\
\qquad\quad x_i, y_i \le 0 \quad (i = 1, 2, 3) &
\end{array}
$$

The optimal solution to this IP is $x_3 = 5$, $y_3 = 3$, $y_2 = 8$, $x_2 = 4$, $x_1 = 0$, $y_1 = 0$, which cannot be correct. According to this solution, $x_1 + x_2 + x_3 = 9$ ads should be placed in IJ. If 9 ads were placed in IJ, however, then it must be that $x_1 = 6$ and $x_2 = 3$. Therefore, we see that the type of formulation used in the Dorian Auto example is correct only if the piecewise linear objective function has a less favorable slope for larger values of x. In our second example, the effectiveness of an ad increased as the number of ads in a magazine increased, and the act of maximizing will not ensure that x_i can be positive only if x_{i-1} assumes its maximum value. In this case, the approach used in the Euing Gas example would yield a correct formulation (see Problem 8).

9.3 The Branch-and-Bound Method for Solving Pure Integer Programming Problems

- Most IPs are solved by using the technique of branch-and-bound.
- Branch-and-bound methods find the optimal solution to an IP by efficiently enumerating the points in a subproblems feasible region.
- Note: If you solve the LP relaxation of a pure IP and obtain a solution in which all variables are integers, then the optimal solution to the LP relaxation is also the optimal solution to the IP.

Example 9 The Telfa Corporation manufactures tables and chairs.

- A table requires 1 hour of labor and 9 square board feet of wood.
- A chair requires 1 hour of labor and 5 square board feet of wood.
- Currently, 6 hours of labor and 45 square board feet of wood are available.
- Each table contributes \$8 to profit, and each chair contributes \$5 to profit.

Formulation of the IP to maximize Telfas profit.

Let

$$
\begin{aligned}
x_1 &= \text{number of tables manufactured} \\
x_2 &= \text{number of chairs manufactured}
\end{aligned}
$$

Because x_1 and x_2 must be integers, Telfa wants to solve the following IP:

$$
\begin{aligned}
\max z = 8x_1 + 5x_2 & \\
\text{s.t.} \quad x_1 + x_2 \le 6 & \qquad \text{(Labor constraint)} \\
9x_1 + 5x_2 \le 45 & \qquad \text{(Wood constraint)} \\
x_1, x_2 \ge 0;\ x_1, x_2 \text{ integer} &
\end{aligned}
$$

Solving the problem

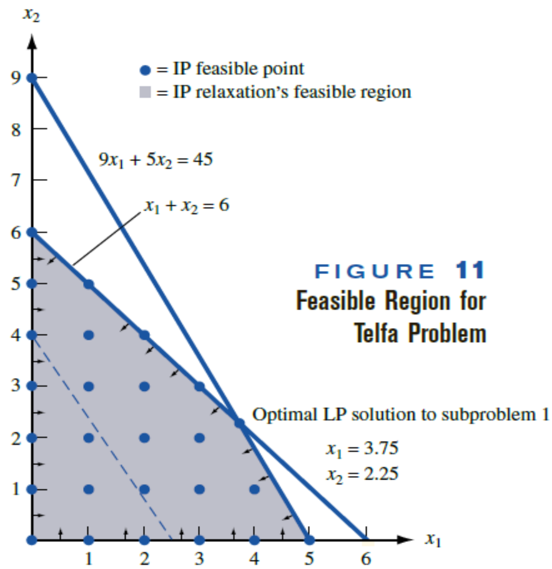


FIGURE 11
Feasible Region for Telfa Problem

- The branch-and-bound method begins by solving the LP relaxation of the IP.
- If all the decision variables assume integer values in the optimal solution to the LP relaxation, then we are done.
- We call the LP relaxation subproblem 1.
- Here the optimal solution to the LP relaxation is $z = 165/4, x_1 = 15/4, x_2 = 9/4$ (see Figure 11).
- From Section 9.1, we have (optimal Z-value for IP) $\leq$ (optimal Z-value for LP relaxation).
- This implies that the optimal z-valuefor the IP cannot exceed 165/4.
- Thus, the optimal z-value for the LP relaxation is an upper bound for Telfas profit.
- We partition the feasible region for the LP relaxation in an attempt to find out more about the location of the IPs optimal solution.
- Choose a variable that is fractional in the optimal solution to the LP relaxation-say, x_1.
- Note that every point in the feasible region for the IP must have either $x_1 \leq 3$ or $x_1 \geq 4$. (Why cant a feasible solution to the IP have $3 < x_1 < 4$?)
- With this in mind, we "branch" on the variable x_1 and create two additional subproblems.
- The optimal solution to subproblem 2 did not yield an all-integer solution.
- Choose a fractional valued variable x_2 in the optimal solution to subproblem 2 and then branch on that variable.
- Partition the feasible region for subproblem 2 into those points having $x_2 \geq 2$ and $x_2 \leq 1$, and get the following two subproblems:

Subproblem 2 Subproblem 1 + Constraint $x_1 \geq 4$.

Subproblem 3 Subproblem 1 + Constraint $x_1 \leq 3$.

- Neither subproblem 2 nor subproblem 3 includes any points with $x_1 = 15/4$.
- The optimal solution to the LP relaxation cannot recur when we solve subproblem 2 or subproblem 3.
- From Figure 12, every point in the feasible region for the Telfa IP is included in the feasible region for subproblem 2 or subproblem 3.
- The feasible regions for subproblems 2 and 3 have no points in common.
- We say that subproblems 2 and 3 were created by branching on x_1.
- Choose any subproblem, say, subproblem 2, that has not yet been solved as an LP.
- From Figure 12, we see that the optimal solution to subproblem 2 is $z = 41, x_1 = 4, x_2 = 9/5$ (point C). See Figure 13.
- A display of all subproblems that have been created is called a **tree**.

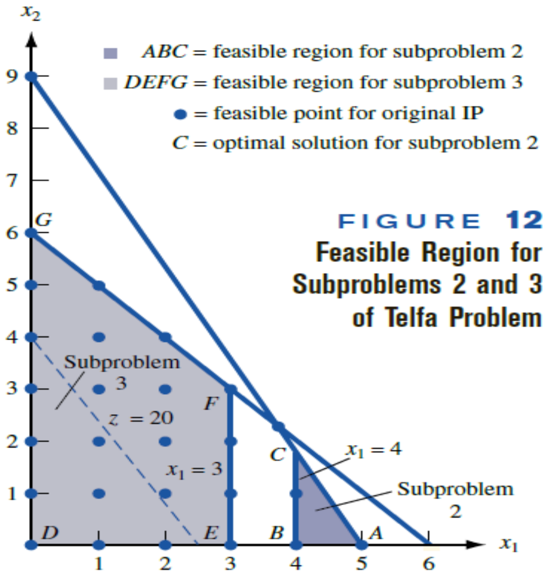
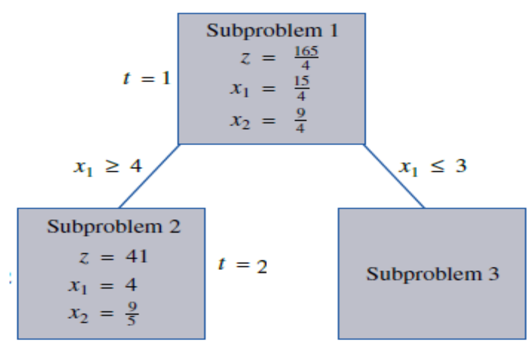


FIGURE 12
Feasible Region for Subproblems 2 and 3 of Telfa Problem

FIGURE 13
Telfa Subproblems 1 and 2 Solved



- Each subproblem is referred to as a **node** of the tree, and each line connecting two nodes of the tree is called an **arc**.
- The constraints associated with any node of the tree are the constraints for the LP relaxation plus the constraints associated with the arcs leading from subproblem 1 to the node.
- The label t indicates the chronological order in which the subproblems are solved.

Subproblem 4

Subproblem 1 + Constraints $x_1 \geq 4$ and $x_2 \geq 2$
$=$ subproblem 2 + Constraint $x_2 \geq 2$.

Subproblem 5

Subproblem 1 + Constraints $x_1 \geq 4$ and $x_2 \leq 1$
$=$ subproblem 2 + Constraint $x_2 \leq 1$.

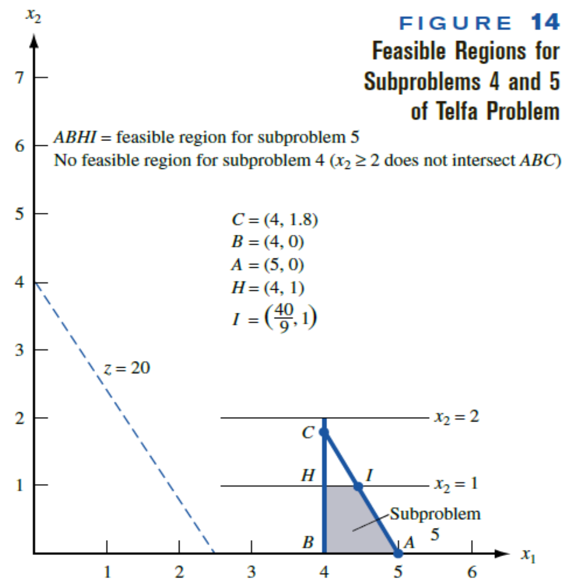
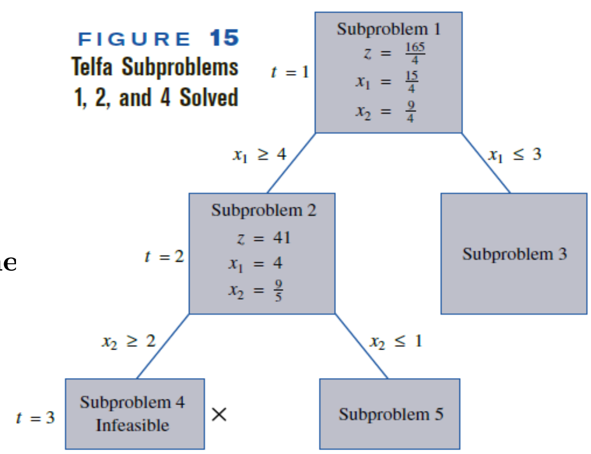


FIGURE 14
Feasible Regions for Subproblems 4 and 5 of Telfa Problem

- The feasible regions for subproblems 4 and 5 are displayed in Figure 14.
- The set of unsolved subproblems consists of subproblems 3, 4, and 5.
- Choose the most recently created subproblem, i.e., subproblem 4 or subproblem 5, to solve. (This is called the LIFO, or last-in-first-out, rule.) Here we choose to solve subproblem 4.
- From Figure 14 we see that subproblem 4 is infeasible. We place an $\times$ by subproblem 4 (see Figure 15).
- We say that the subproblem (or node) is **fathome** (No need to branch out anymore.) See Figure 15.
- Now the only unsolved subproblems are subproblems 3 and 5. We consider subprogram 5 by the LIFO rule.

FIGURE 15
Telfa Subproblems 1, 2, and 4 Solved



- From Figure 14, we see that the optimal solution to subproblem 5 is point I in Figure 14: $z = 365/9, x_1 = 40/9, x_2 = 1$.
- So we choose to partition subproblem 5s feasible region by branching on the fractional-valued variable x_1 two new subproblems

Subproblem 6 Subproblem 5 + Constraint $x_1 \geq 5$.

Subproblem 7 Subproblem 5 + Constraint $x_1 \leq 4$.

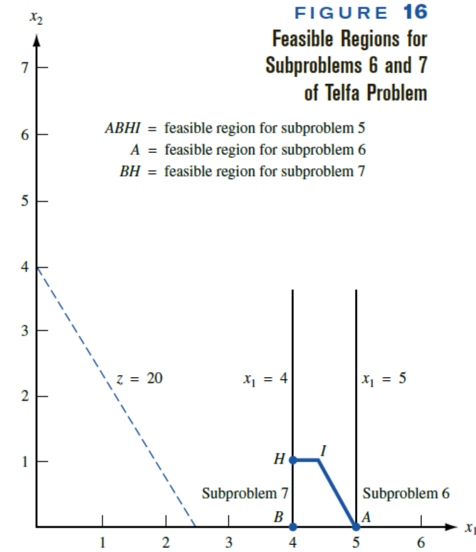
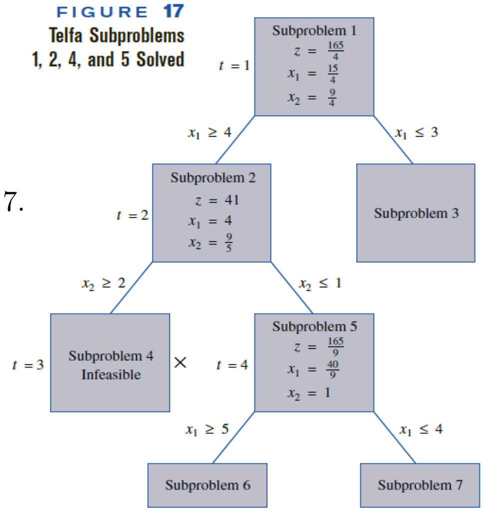


FIGURE 16
Feasible Regions for Subproblems 6 and 7 of Telfa Problem

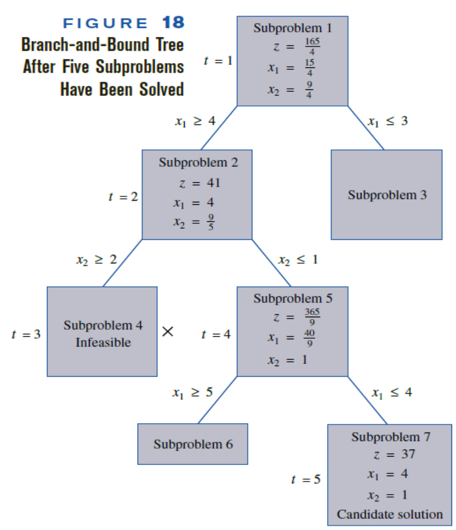
- Subproblems 6 and 7 include all integer points that were included in the feasible region for subproblem 5.
- No point having $x_1 = 40/9$ can be in the feasible region for subproblem 6 or subproblem 7.
- The optimal solution to subproblem 5 will not recur when we solve subproblems 6 and 7.
- Our tree now looks as shown in Figure 17.

FIGURE 17
Telfa Subproblems 1, 2, 4, and 5 Solved



- Subproblems 3, 6, and 7 are now unsolved.
- The LIFO rule implies that we next solve subproblem 6 or subproblem 7. We solve subproblem 7.
- From Figure 16, we see that the optimal solution to subproblem 7 is point $H: z = 37, x_1 = 4, x_2 = 1$.
- Both x_1 and x_2 assume integer values, so this solution is feasible for the original IP.
- We now know that subproblem 7 yields a feasible integer solution with $z = 37$.
- We also know that subproblem 7 cannot yield a feasible integer solution having $z = 37$.
- Thus, further branching on subproblem 7 will yield no new information about the optimal solution to the IP, and subproblem has been fathomed.
- The tree to date is pictured in Figure 18.

FIGURE 18
Branch-and-Bound Tree After Five Subproblems Have Been Solved



- A solution obtained by solving a subproblem in which all variables have integer values is a candidate solution.
- Because the candidate solution may be optimal, we must keep a candidate solution until a better feasible solution to the IP (if any exists) is found.
- We have a feasible solution to the original IP with $z = 37$, so the optimal z-value for the IP is 37.
- Thus, the z-value for the candidate solution is a lower bound on the optimal z-value for the original IP.
- We note this by placing the notation LB is 37 in the box corresponding to the next solved subproblem (see Figure 19).
- The only remaining unsolved subproblems are 6 and 3.
- Following the LIFO rule, we next solve subproblem 6.
- From Figure 16, we find that the optimal solution to subproblem 6 is point $A : z = 40, x_1 = 5, x_2 = 0$.
- All decision variables have integer values, so this is a candidate solution.
- Its z-value of 40 is larger than the z-value of the best previous candidate (candidate 7 with $z = 37$).

FIGURE 19
Branch-and-Bound Tree After Six Subproblems Have Been Solved

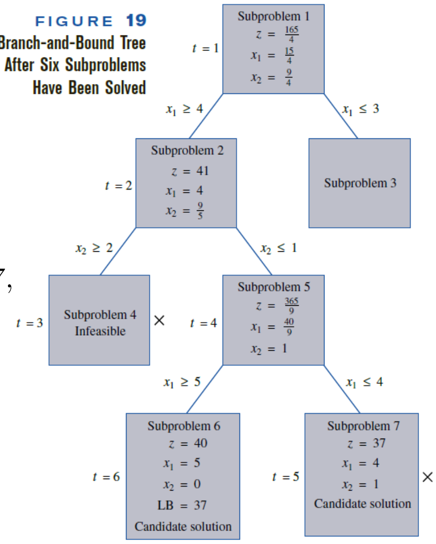
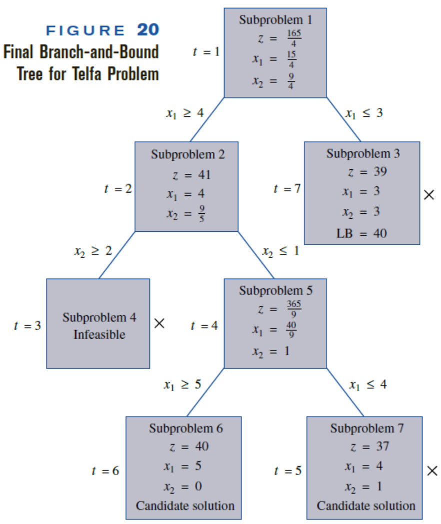


FIGURE 20
Final Branch-and-Bound Tree for Telfa Problem



- Thus, subproblem 7 cannot yield the optimal solution of the IP (we denote this fact by placing an $\times$ by subprblem 7). We also update our LB to 40. (See Figure 20).
- Subproblem 3 is the only remaining unsolved problem.
- From Figure 12, the optimal solution to subproblem 3 is point $F : z = 39, x_1 = x_2 = 3$.
- Subproblem 3 cannot yield a z-value exceeding the current lower bound of 40, so it cannot yield the optimal solution to the original IP.
- Therefore, we place an $\times$ by it in Figure 20. From Figure 20, there are no remaining unsolved subproblems, and that only subproblem 6 can yield the optimal solution to the IP.
- Thus, the optimal solution to the IP is for Telfa to manufacture 5 tables and 0 chairs.
- This solution will contribute \$40 to profits.

- In using the branch-and-bound method to solve the Telfa problem, we have implicitlyenumerated all points in the IPs feasible region.
- Eventually, all such points (except for the optimal solution) are eliminated from consideration, and the branch-and-bound procedure is complete.
- To show that the branch-and-bound procedure actually does consider all points in the IPs feasible region, we examine several possible solutions to the Telfa problem and show how the procedure found these points to be nonoptimal.
- For example, how do we know that $x_1 = 2, x_2 = 3$ is not optimal?
- This point is in the feasible region for subproblem 3, and we know that all points in the feasible region for subproblem 3 have $z = 39$.
- Thus, our analysis of subproblem 3 shows that $x_1 = 2, x_2 = 3$ cannot beat $z = 40$ and cannot be optimal.
- As another example, why isnt $x_1 = 4, x_2 = 2$ optimal?
- Following the branches of the tree, we find that $x_1 = 4, x_2 = 2$ is associated with subproblem 4.
- Because no point associated with subproblem 4 is feasible, $x_1 = 4, x_2 = 2$ must fail to satisfy the constraints for the original IP and thus cannot be optimal for the Telfa problem.
- In a similar fashion, the branch-and-bound analysis has eliminated all points x_1, x_2 (except for the optimal solution) from consideration.
- For the simple Telfa problem, the use of the branch-and-bound method may seem like using a cannon to kill a fly.
- But for an IP in which the feasible region contains a large number of integer points, the procedure can be very efficient for eliminating nonoptimal points from consideration.
- For example, suppose we are applying the branch-and-bound method and our current LB is 42.
- Suppose we solve a subproblem that contains 1 million feasible points for the IP.
- If the optimal solution to this subproblem has $z = 42$, then we have eliminated 1 million nonoptimal points by solving a single LP!
- The key aspects of the branch-and-bound method for solving pure IPs (mixed IPs are considered in the next section) may be summarized as follows:

Step 1 If it is unnecessary to branch on a subproblem, then it is fathomed. The following three situations result in a subproblem being fathomed:

(1) The subproblem is infeasible;

(2) the subproblem yields an optimal solution in which all variables have integer values; and

(3) the optimal z-value for the subproblem does not exceed (in a max problem) the current LB.

Step 2 A subproblem may be eliminated from consideration in the following situations:

(1) The subproblem is infeasible (in the Telfa problem, subproblem 4 was eliminated for this reason);

(2) the LB (representing the z-value of the best candidate to date) is at least as large as the z-value for the subproblem (in the Telfa problem, subproblems 3 and 7 were eliminated for this reason).

Remarks

1. For some IPs, the optimal solution to the LP relaxation will also be the optimal solution to the IP. Suppose the constraints of the IP are written as $Ax = b$. If the determinant of every square submatrix of A is $1, -1$, or 0, we say that the matrix A is **unimodular**.

   If A is unimodular and each element of b is an integer, then the optimal solution to the LP relaxation will assign all variables integer values and will therefore be the optimal solution to the IP.

   For example, the constraint matrix of any MCNFP is unimodular. Thus, as was discussed in Chapter 8, any MCNFP in which each nodes net outflow and each arcs capacity are integers will have an integer-valued solution.

2. As a general rule, the more an IP looks like an MCNFP, the easier the problem is to solve by branch-and-bound methods. Thus, in formulating an IP, it is good to choose a formulation in which as many variables as possible have coefficients of $1, -1$, and 0.

   To illustrate this idea, recall that the formulation of the Nickles (lockbox) problem given in Section 9.2 contained 16 constraints of the following form:

   Formulation 1 $\quad x_{ij} \leq y_j \quad (i = 1, 2, 3, 4; j = 1, 2, 3, 4)$.

   As we have already seen in Section 9.2, if the 16 constraints in above are replaced by the following 4 constraints, then an equivalent formulation results:

   Formulation 2

   $$\begin{array}{ll} x_{11} + x_{21} + x_{31} + x_{41} \leq 4y_1 & \qquad x_{12} + x_{22} + x_{32} + x_{42} \leq 4y_2 \\ x_{13} + x_{23} + x_{33} + x_{43} \leq 4y_3 & \qquad x_{14} + x_{24} + x_{34} + x_{44} \leq 4y_2 \end{array}$$

   Because formulation 2 has $16 - 4 = 12$ fewer constraints than formulation 1, one might think that formulation 2 would require less computer time to find the optimal solution. This turns out to be untrue.

   Reason. The feasible region of the LP relaxation of formulation 2 contains many more non-integer points than the feasible region of formulation 1.

   For example, the point $y_1 = y_2 = y_3 = y_4 = 1/4$, $x_{11} = x_{22} = x_{33} = x_{44} = 1$ and $x_{ij} = 0$ for other i, j is in the feasible region for the LP relaxation of formulation 2, but not for formulation 1.

3. When solving an IP in the real world, we are usually happy with a near-optimal solution. For example, if we can find a feasible solution with z-value closed to the LP relaxation, say, the ratio of the two optimal values is larger than .9, then we might want to save the effort, computer time, real time for making decision, and just use this feasible solution.

   For this reason, the branch-and-bound procedure is often terminated when a candidate solution is found with a z-value close to the z-value of the LP relaxation.

4. Subproblems for branch-and-bound problems are often solved using some variant of the dual simplex algorithm because we always add a constraint to generate new subprogram.

5. Sometimes, one can show that if the constraints $x_k = i$ and $x_k = i + 1$ are added, then the optimal solution to the first subproblem will have $x_k = i$ and the optimal solution to the second subproblem will have $x_k = i + 1$. This observation is very helpful, especially, when we graphically solve subproblems.

9.4 The Branch-and-Bound Method for Mixed Integer Programming Problems

- In a mixed IP, some variables are required to be integers and others are allowed to be either integers or nonintegers.
- To solve a mixed IP by the branch-and-bound method, modify the method described in Section 9.3 by branching only on variables that are required to be integers.

Example $\max Z = 2x_1 + x_2$

s.t. $5x_1 + 2x_2 = 8$

$x_1 + x_2 \le 3$

$x_1, x_2 \ge 0$; x_1 integer

- We begin by solving the LP relaxation of the IP.
- The optimal solution of the LP relaxation is $Z = 11/3, x_1 = 2/3, x_2 = 7/3$.
- Because x_2 is allowed to be fractional, we do not branch on x_2.
- So, we branch at x_1 and solve subproblem 2.
- The optimal solution to subproblem 2 is the candidate solution $z = 3, x_1 = 0, x_2 = 3$.

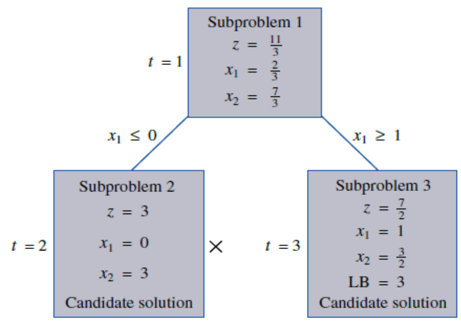


FIGURE 21
Branch-and-Bound Tree for Mixed IP

- We solve subproblem 3 and obtain the candidate solution $z = 7/2, x_1 = 1, x_2 = 3/2$.
- The z-value from the subproblem 3 candidate exceeds the z-value for the subproblem 2 candidate, so subproblem 2 can be eliminated from consideration, and the subproblem 3 candidate $(z = 7/2, x_1 = 1, x_2 = 3/2)$ to the mixed IP.

Solving Knapsack Problems by the Branch-and-Bound Method

- A knapsack problem is an IP with a single constraint, and each variable must equal 0 or 1.
- So, a knapsack problem may be written as

$$\max z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$
$$\text{s.t.} \quad a_1x_1 + a_2x_2 + a_nx_n \leq b$$
$$x_i \in \{0,1\} \quad (i = 1, 2, ..., n).$$

- Recall that c_i is the benefit obtained if item i is chosen, b is the amount of an available resource, and a_i is the amount of the available resource used by item i.
- When knapsack problems are solved by the branch-and-bound method, two aspects of the method greatly simplify.

(1) Because x_i must equal 0 or 1, branching on x_i will yield an $x_i = 0$ and an $x_i = 1$ branch.

(2) Also, the LP relaxation (and other subproblems) may be solved by inspection.

Reason. a_i/c_i may be interpreted as the benefit item i earns for each unit of the resource used by item i. Thus, the best (worst) items have the largest (smallest) values of a_i/c_i. So, one can choose items with large c_i/a_i values.

Example Consider the problem:

$$\max z = 40x_1 + 80x_2 + 10x_3 + 10x_4 + 4x_5 + 20x_6 + 60x_7$$
$$\text{s.t. } 40x_1 + 50x_2 + 30x_3 + 10x_4 + 10x_5 + 40x_6 + 30x_7 \leq 100$$
$$x_i \in \{0,1\} \quad (i = 1, 2, ..., 7).$$

We begin by computing the a_i/c_i ratios and ordering the variables from best to worst.

i	1	2	3	4	5	6	7
c_i/a_i	1	8/5	1/3	1	4/10	1/2	2

Then choose item 7, and $100 - 30 = 70$ units of the resource remain.
Then choose 2, and $70 - 50 = 20$ units of the resource remain.
Then choose item 4 or item 1; we choose item 4, and $20 - 10 = 10$ units of the resource remain.
The best remaining item is item 1. We fill the knapsack with as much of item 1 as we can.
Because only 10 units of the resource remain, we set $x_1 = 1/4$ (for the LP relaxation problem).
Thus an optimal solution to the LP relaxation is $z = 80 + 60 + 10 + (1/4)40 = 160$ with
$x_7 = x_2 = x_4 = 1$ and $x_1 = 1/4$.

Example 1 Stockco capital budgeting problem

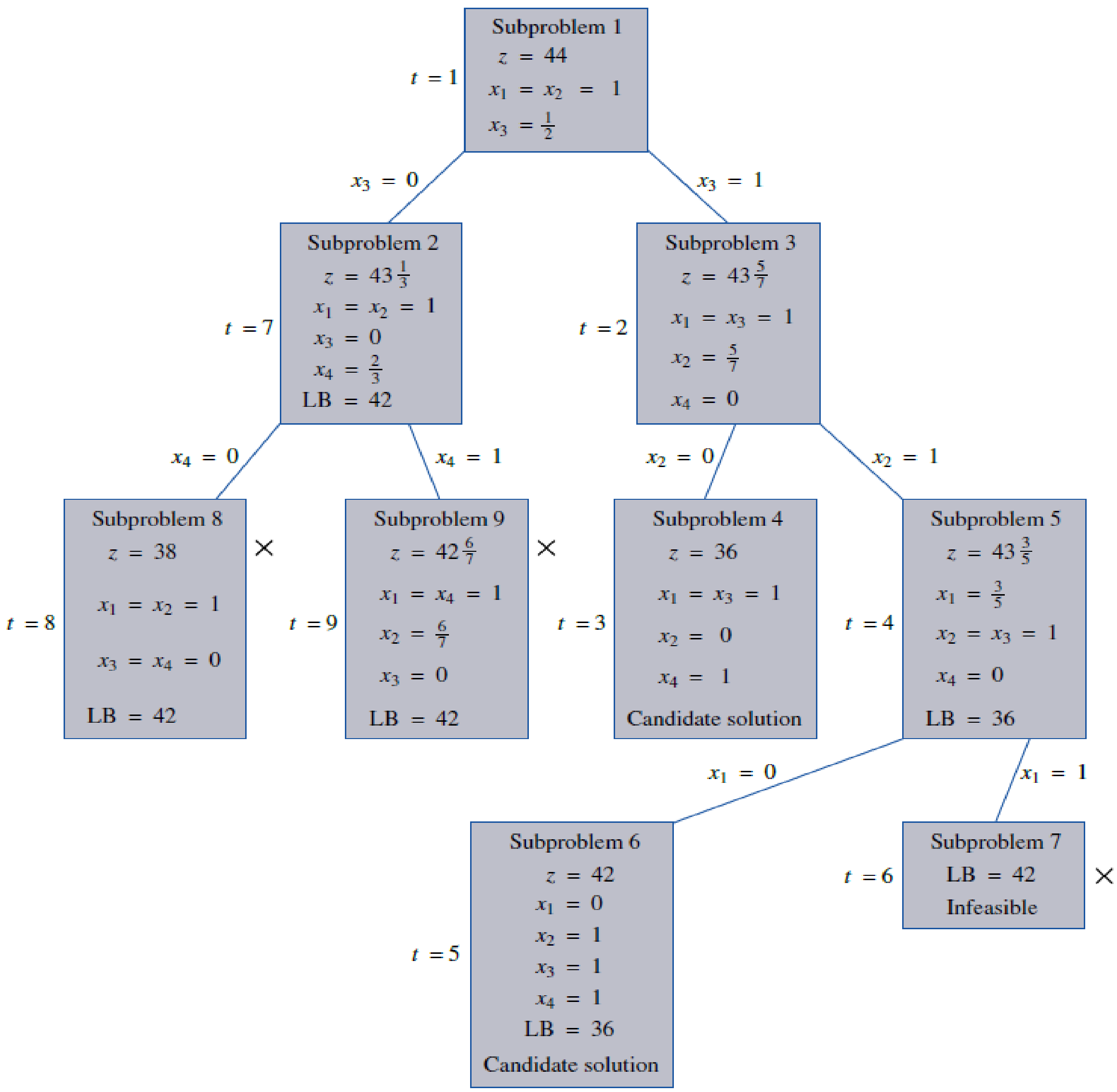
$$\max z = 16x_1 + 22x_2 + 12x_3 + 8x_4$$

$$\text{s.t.} \qquad 5x_1 + 7x_2 + 4x_3 + 3x_4 \leq 14$$

$$x_1, \ldots, x_4 \in \{0, 1\}.$$

The branch-and-bound tree for this problem is shown in the following Figure.

The optimal solution is $z = 42$, with $x_1 = 0, x_2 = x_3 = x_4 = 1$.



9.6 Solving Combinatorial Optimization Problems

Branch-and-Bound Machine Scheduling

- Four jobs must be processed on a single machine.
- The time required to process each job and the date the job is due are shown in Table 63.

TABLE 63
Durations and Due Date of Jobs

Job	Days Required to Complete Job	Due Date
1	6	End of day 8
2	4	End of day 4
3	5	End of day 12
4	8	End of day 16

- The delay of a job is the number of days after the due date that a job is completed.
- In what order should the jobs be processed to minimize the total delay of the four jobs?
- Suppose the jobs are processed in the following order: job 1, job 2, job 3, and job 4.
  By Table 64, the total delay is $0 + 6 + 3 + 7 = 16$ days.

TABLE 64
Delays Incurred If Jobs Are Processed in the Order 1-2-3-4

Job	Completion Time of Job	Delay of Job
1	6	0
2	6 + 4 = 10	10 − 4 = 6
3	6 + 4 + 5 = 15	15 − 12 = 3
4	6 + 4 + 5 + 8 = 23	23 − 16 = 7

- Consider the branch-and-bound approach.
- Let x_{ij} be 1 if job i is scheduled to take the jth slot to be processed.
- Partition all solutions according to the job that is **last** processed, i.e., $x_{14} = 1$, $x_{24} = 1$, $x_{34} = 1$, or $x_{44} = 1$.
- We create a node by branching, we obtain a lower bound on the total delay (D) associated with the node.
- So, if $x_{44} = 1$, then job 4 is the last job to be processed, job 4 will be completed at the end of day $6 + 4 + 5 + 8 = 23$ and will be $23 - 16 = 7$ days late.

TABLE 63
Durations and Due Date of Jobs

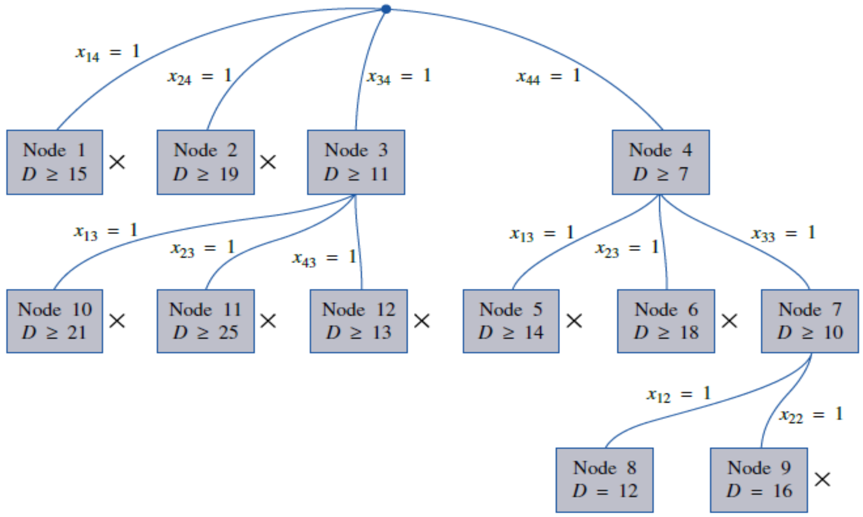
Job	Days Required to Complete Job	Due Date
1	6	End of day 8
2	4	End of day 4
3	5	End of day 12
4	8	End of day 16

TABLE 64
Delays Incurred If Jobs Are Processed in the Order 1-2-3-4

Job	Completion Time of Job	Delay of Job
1	6	0
2	6 + 4 = 10	10 − 4 = 6
3	6 + 4 + 5 = 15	15 − 12 = 3
4	6 + 4 + 5 + 8 = 23	23 − 16 = 7

- Similar reasoning applies to other nodes.

FIGURE 23



- We use the **jumptracking** approach instead of the LIFO (a.k.a. backtracking approach) to branch on the nodes. So, we branch at node 3 and node 4.

- Any job sequence associated with node 4 must have $x_{13} = 1, x_{23} = 1$, or $x_{33} = 1$.

- Branching on node 4 yields nodes 57 in Figure 23.

- For node 7, job 4 will be delayed by 7 days, and job 3 will be the third job processed and completed after $6 + 4 + 5 = 15$ causing a delay of $15 - 12 = 3$ days.

- Any sequence associated with node 7 must have $D \geq 7 + 3 = 10$ days delay.

- Branching at node 7, we arrive at node 8 and 9.

- Now we can eliminate all other nodes using node 8.

- So, the optimal solution is the job sequence $2 - -1 - -3 - -4$ with a total delay of 12 days.

Example 11 Traveling Salesman Problem

- Joe State lives in Gary, Indiana. He owns insurance agencies in Gary, Fort Wayne, Evansville, Terre Haute, and South Bend.
- Each December, he visits each of his insurance agencies.
- The distance between each agency (in miles) is shown in Table 65.

TABLE 65
Distance between Cities in Traveling Salesperson Problem

Day	Gary	Fort Wayne	Evansville	Terre Haute	South Bend
City 1 Gary	0	132	217	164	58
City 2 Fort Wayne	132	0	290	201	79
City 3 Evansville	217	290	0	113	303
City 4 Terre Haute	164	201	113	0	196
City 5 South Bend	58	79	303	196	0

- What order of visiting his agencies will minimize the total distance traveled?
- Joe must determine the order of visiting the five cities that minimizes the total distance traveled.
- For example, Joe could choose to visit the cities in the order 134521. Then he would travel a total of $217 + 113 + 196 + 79 + 132 = 737$ miles.
- Let $x_{ij} = 1$ if Joe leaves the ith city, and go to the jth city.
- Let c_{ij} be the distance from the ith city to the jth city for $i \neq j$, and $c_{ii} = M$ for a big M.
- We want to find a low cost solution with $x_{i_1,i_2}, x_{i_2,i_3}, \ldots, x_{i_5,i_1}$ so that $\{i_1, i_2, i_3, i_4, i_5\} = \{1, 2, 3, 4, 5\}$.
- An itinerary that begins and ends at the same city and visits each city once is called a **tour**.
- One may try to solve an assignment problem to get the solution of the TSL.
- We set it up as subproblem 1.
- If the solution of subproblem 1 is a tour, then it is an optimal solution for the TSL.
- But the solution may not be a tour.
- For example, the optimal solution to the assignment problem might be $x_{15} = x_{21} = x_{34} = x_{43} = x_{52} = 1$.
- We will use a branch and bound method.
- The above solution contains two subtours (1521 and 343).
- We will try to exclude the subtour $3 - 4 - 3$ by considering two subproblems.

Subproblem 2

Subproblem 1 + ($x_{34} = 0$, or $c_{34} = M$).

Subproblem 3

Subproblem 1 + ($x_{43} = 0$, or $c_{43} = M$).

- Apply the Hungarian method to the cost matrix in Table 67.
- The optimal solution to subproblem 2 is $z = 652$, $x_{14} = x_{25} = x_{31} = x_{43} = x_{52} = 1$.
- This solution includes the subtours 1431 and 252, so this cannot be the optimal solution.
- Now branch on subproblem 2 in an effort to exclude the subtour 252.
- To ensure that either x_{25} or x_{52} equals zero. Thus, we add the following subproblems:

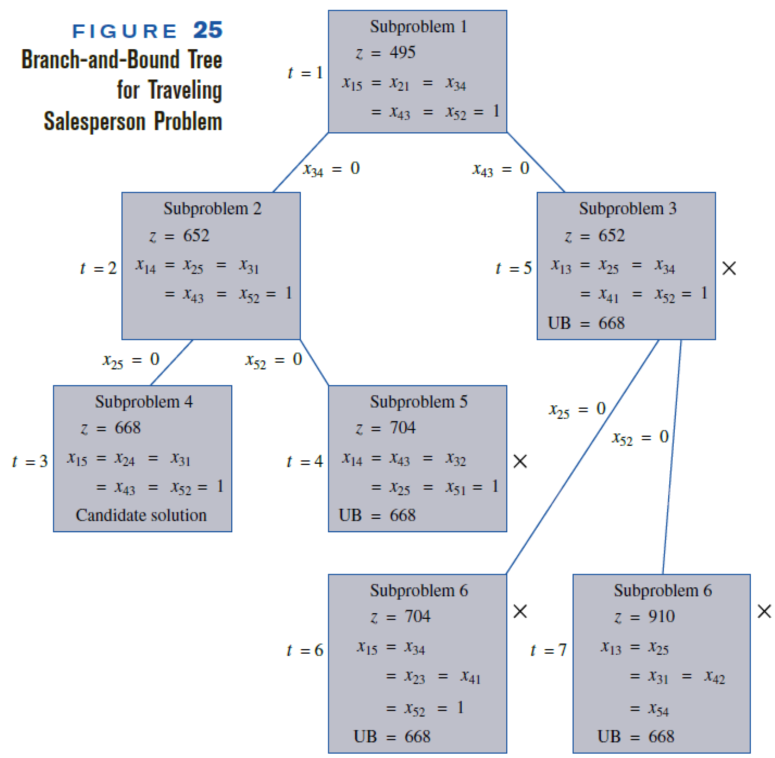
TABLE 66
Cost Matrix for Subproblem 1

	City 1	City 2	City 3	City 4	City 5
City 1	M	132	217	164	58
City 2	132	M	290	201	79
City 3	217	290	M	113	303
City 4	164	201	113	M	196
City 5	58	79	303	196	M

TABLE 67
Cost Matrix for Subproblem 2

	City 1	City 2	City 3	City 4	City 5
City 1	M	132	217	164	58
City 2	132	M	290	201	79
City 3	217	290	M	M	303
City 4	164	201	113	M	196
City 5	58	79	303	196	M

FIGURE 25
Branch-and-Bound Tree for Traveling Salesperson Problem



Subproblem 4

Subproblem 2 $+(x_{25} = 0$, or $c_{25} = M)$.

Subproblem 5

Subproblem 2 $+(x_{52} = 0$, or $c_{52} = M)$.

- Following the LIFO approach, we solve subproblem 4 or subproblem 5.
- Consider subproblem 4 and apply the Hungarian method (Table 68); the optimal solution is $z = 668, x_{15} = x_{24} = x_{31} = x_{43} = x_{52} = 1$.
- This solution contains no subtours and yields the tour 152431.
- Thus, subproblem 4 yields a solution with $z = 668$.
- We then solve subproblem 5 by Hungarian method (Table 69). The optimal solution to subproblem 5 is $z = 704, x_{14} = x_{43} = x_{32} = x_{25} = x_{51} = 1$.
- This solution is a tour, but $z = 704$. The node is fathomed.
- Only subproblem 3 remains; the optimal solution in Table 70 is $x_{13} = x_{25} = x_{34} = x_{41} = x_{52} = 1, z = 652$.
- This solution contains the subtours 1341 and 252. Because $652 < 668$, it is still possible for subproblem 3 to yield a solution with no subtours with $z < 668$.
- Thus, we now branch on subproblem 3 in an effort to exclude the subtours.

TABLE 68
Cost Matrix for Subproblem 4

	City 1	City 2	City 3	City 4	City 5
City 1	M	132	217	164	58
City 2	132	M	290	201	M
City 3	217	290	M	M	303
City 4	164	201	113	M	196
City 5	58	79	303	196	M

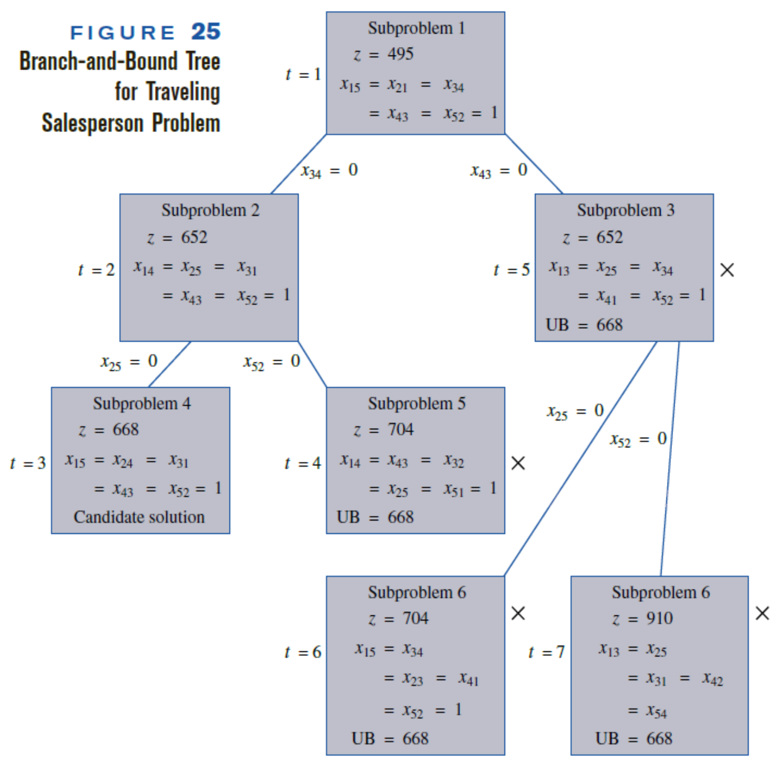
TABLE 69
Cost Matrix for Subproblem 5

	City 1	City 2	City 3	City 4	City 5
City 1	M	132	217	164	58
City 2	132	M	290	201	79
City 3	217	290	M	M	303
City 4	164	201	113	M	196
City 5	58	M	303	196	M

TABLE 70
Cost Matrix for Subproblem 3

	City 1	City 2	City 3	City 4	City 5
City 1	M	132	217	164	58
City 2	132	M	290	201	79
City 3	217	290	M	113	303
City 4	164	201	M	M	196
City 5	58	79	303	196	M

FIGURE 25
Branch-and-Bound Tree for Traveling Salesperson Problem



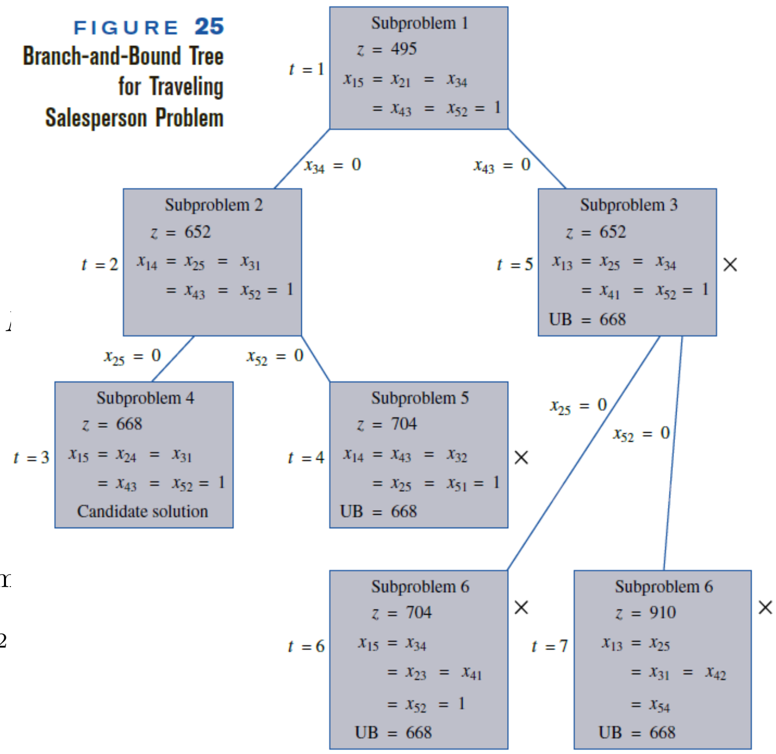
- Any feasible solution to the traveling salesperson problem that emanates from subproblem 3 must have either $x_{25} = 0$ or $x_{52} = 0$.
- So we create subproblems 6 and 7.

Subproblem 6

Subproblem 3 $+(x_{25} = 0$, or $c_{25} = M)$.

Subproblem 7 Subproblem 3 $+(x_{52} + 0$, or $c_{52} =$

FIGURE 25
Branch-and-Bound Tree for Traveling Salesperson Problem



- Solve subproblem 6. The optimal solution is $x_{15} = x_{34} = x_{23} = x_{41} = x_{52} = 1, z = 704$. This node is fathomed.
- The only remaining subproblem is subproblem The optimal solution is $x_{13} = x_{25} = x_{31} = x_{42}$ This node is fathomed.
- So, subproblem 4 yields the optimal solution: Joe should travel from Gary to South Bend, from South Bend to Fort Wayne, from Fort Wayne to Terre Haute, from Terre Haute to Evansville, and from Evansville to Gary. Joe will travel a total distance of 668 miles.

An integer programming formulation for TSP

We now discuss how to formulate an IP whose solution will solve a TSP. We note, however, that the formulation of this section becomes unwieldy and inefficient for large TSPs. Suppose the TSP consists of cities $1, 2, 3, \ldots, N$. For $i \neq j$ let c_{ij} = distance from city i to city j and let $c_{ii} = M$, where M is a very large number (relative to the actual distances in the problem). Setting $c_{ii} = M$ ensures that we will not go to city i immediately after leaving city i. Also define

$$x_{ij} = \begin{cases} 1 & \text{if the solution to TSP goes from city } i \text{ to city } j \\ 0 & \text{otherwise} \end{cases}$$

Then the solution to a TSP can be found by solving

$$\min z = \sum_i \sum_j c_{ij} x_{ij} \tag{40}$$

$$\text{s.t.} \quad \sum_{i=1}^{N} x_{ij} = 1 \quad (\text{for } j = 1, 2, \ldots, N) \tag{41}$$

$$\sum_{j=1}^{N} x_{ij} = 1 \quad (\text{for } i = 1, 2, \ldots, N) \tag{42}$$

$$u_i - u_j + N x_{ij} \leq N - 1 \quad (\text{for } i \neq j;\ i = 2, 3, \ldots, N;\ j = 2, 3, \ldots, N) \tag{43}$$

$$\text{All } x_{ij} = 0 \text{ or } 1, \text{ All } u_j \geq 0$$

- The objective function (40) gives the total length of the arcs included in a tour.
- The constraints in (41) ensure that we arrive once at each city.
- The constraints in (42) ensure that we leave each city once.
- The constraints in (43) are the key to the formulation. They ensure the following:

1) Any set of x_{ij}s containing a subtour will violate (43).

2) Any set of x_{ij}s that forms a tour will be feasible, i.e., there is a set of u_js satisfying (43).

Example Consider the assignment of the five vertex network that contains the subtours

$1-5-2-1$ and $3-4-3$.

We obtain $u_3 - u4 + 5x_{34} \leq 4$ and $u_4 - u_3 + 5x_{43} \leq 4$.

Adding these constraints yields $5(x_{34} + x_{43}) \leq 8$.

Clearly, this rules out the possibility that $x_{43} + x_{34} = 1$, or any subtour.

Next, we show that if the assignment does not contain a subtour then it is feasible.

Consider the tour $1-3-4-5-2-1$.

Then we choose $u_1 = 1, u_3 = 2, u_4 = 3, u_5 = 4, u_2 = 5$.

Consider any constraint corresponding to an arc having $x_{ij} = 1$.

For example, the constraint corresponding to x_{52} is $u_5 - u_2 + 5x_{52} \leq 4$.

Because $u_5 - u_2 = -1$, the constraint for x_{52} in (43) reduces to $-1 + 5 \leq 4$.

Now consider a constraint corresponding to an $x_{ij} = 0$, say, x_{32}.

We obtain the constraint $4 \geq u_3 - u_2 + 5x_{32} = u_3 - u_2$, which is at most $5 - 1 = 4$.

9.7 Implicit Enumeration

- Note that an integer constraint on $x < 2^{n+1}$ by setting $x = 2^n u_n + \cdots + u_0$ with $u_i \in \{0, 1\}$.
- We will see that 0-1 problems are easier to solve.
- The tree used in the implicit enumeration method is similar to those used to solve 0-1 knapsack problems.
- At each node, the values of some of the variables are specified.
- For instance, suppose a 0-1 problem has variables $x_1, x_2, x_3, x_4, x_5, x_6$.
- Part of the tree looks like Figure 26.
- At node 4, the values of x_3, x_4, and x_2 are specified.
- These variables are referred to as **fixed** variables.
- All variables whose values are unspecified at a node are called free variables.
- Thus, at node 4, x_1, x_5, and x_6 are **free** variables.
- For any node, a specification of the values of all the free variables is called a **completion** of the node.
- Thus $x_1 = 1, x_5 = 1, x_6 = 0$ is a completion of node 4.

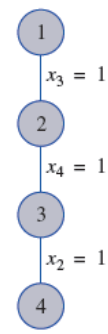


FIGURE 26

Some general ideas:

1. At a node, find a completion of the free variables that optimize the objective function.

   If it is feasible, then we have the optimal solution at the node and no branching is needed.

   For example, $\max z = 4x_1 + 2x_2 - x_3 + 2x_4$

   s.t. $x_1 + 3x_2 - x_3 - 2x_4 \geq 1$,

   $x_i \in \{0, 1\}, \quad i = 1, 2, 3, 4.$

   If $(x_1, x_2, x_3, x_4) = (0, 1, 0, 1)$ is given, we see that the solution is feasible. The node is fathomed with $z = 4$.

2. If an optimal completion is not feasible, then we have an upper bound for the completion of the rest of the node.

3. If an optimal completion is worse than a feasible solution found before, the node can be removed.

4. If there is a constraint such that the most feasible completion fails, then there is no completion that are feasible.

Example: Implicit enumeration $\max z = -7x_1 - 3x_2 - 2x_3 - x_4 - 2x_5$

$$\begin{aligned} \text{s.t.} \quad & -4x_1 - 2x_2 + x_3 - 2x_4 - x_5 \le -3 && (47)\\ \text{s.t.} \quad & -4x_1 - 2x_2 - 4x_3 + x_4 + 2x_5 \le -7 && (48)\\ & x_i \in \{0,1\} \quad i = 1,2,3,4,5. \end{aligned}$$

- At node 1, all variables are free.
- The optimal solution $(x_1, x_2, x_3, x_4, x_5) = (0,0,0,0,0)$ is not feasible.
- Then check feasibility for (47), let $(x_1, x_2, x_3, x_4, x_5) = (1,1,0,1,1)$
- For (48), one may let $(x_1, x_2, x_3, x_4, x_5) = (1,1,1,0,0)$
- We set node 2 and node 3 for $x_1 = 1$ and 0, respectively.
- Consider node 2. The best completion is $(1,0,0,0,0)$, which is infeasible.
- Check $(1,1,0,1,1)$ and $(1,1,1,0,0)$, (47) and (48) are feasible.
- Branch on node 2 to get node 4 and node 5 with $x_2 = 1$ and 0, resp.
- Check optimal solution, feasible solutions at node 5; branch out to node
- Use the LIFO rule, consider node 6.
- The best completion is $(1,0,1,0,0)$, which is feasible.

  We found a candidate solution with $z = -9$.
- By the LIFO rule, consider node 7.

  The best completion is $(1,0,0,0,0)$ with $z = -7$, which is better than $z = -9$.
- We check node 7 to see whether it has any feasible completion.

  For (47), $(1,0,0,1,1)$ is feasible; for (48), $(1,0,0,0,0)$ is infeasible
- Thus, no completion of node 7 can satisfy (48), and the node can be eliminated (Figure 30).

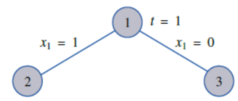


FIGURE 27

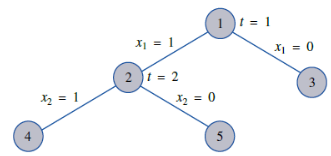


FIGURE 28

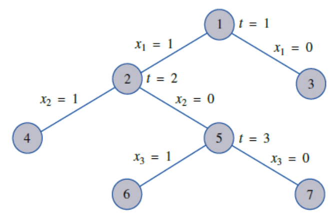


FIGURE 29

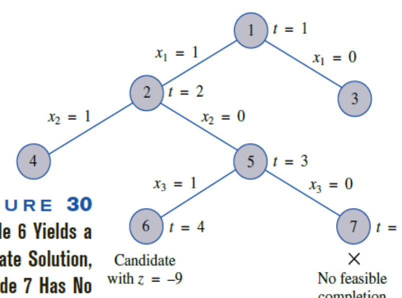


FIGURE 30
Node 6 Yields a Candidate Solution, and Node 7 Has No Feasible Completion

- By the LIFO rule, we analyze node 4.
- The best completion is $(1,1,0,0,0)$ with $z=-10$.
- Thus, node 4 cannot beat the previous candidate solution from node 6, where $z=-9$.
- Then node 4 may be eliminated; see Figure 31.
- It remains to consider node 3.
- The best completion is $(0,0,0,0,0)$, which is impossible.
- Because $z=0$, it is possible that node 3 can yield better solution.
- We now check whether node 3 has any feasible completion:

  $(0,1,1,1,1)$ satisfies (47), but $(0,1,1,0,0)$ fails (48).
- So, node 3 may be eliminated from consideration.
- We now have the tree in Figure 32.
- Because there are no nodes left to analyze, the node 6 with

  $(x_1,x_2,x_3,x_4,x_5)=(1,0,1,0,0)$ and $z=-9$ is optimal.

FIGURE 31 Node 4 Cannot Beat Node 6 Candidate

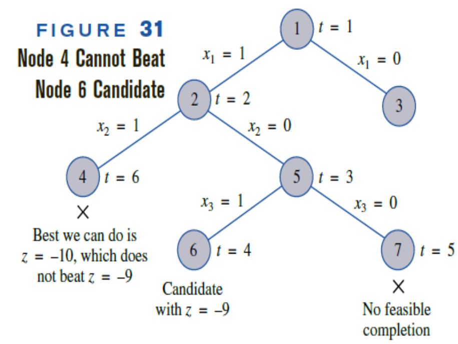
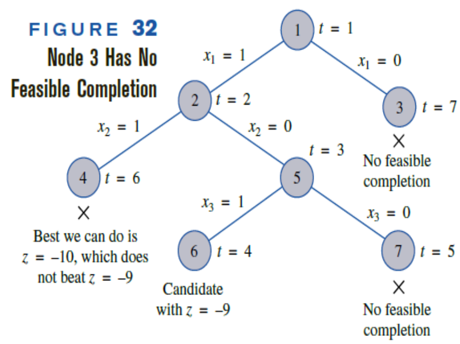


FIGURE 32 Node 3 Has No Feasible Completion



Cutting plane algorithm

Example $\max z = 8x_1 + 5x_2$

s.t. $x_1 + x_2 \leq 6$

s.t. $9x_1 + 5x_2 \leq 45$

$x_1, x_2 \geq 0,\ x_1, x_2$ integer.

We use slack variable and get the optimal solution for the relaxation problem $z = 41.25$ with $(x_1, x_2) = (3.75, 2.25)$.

TABLE 83
Optimal Tableau for LP Relaxation of Telfa

z	x_1	x_2	s_1	s_2	rhs
0	0	1	2.25	−0.25	2.25
0	1	0	−1.25	0.25	3.75
1	0	0	1.25	0.75	41.25

Select a variable in fractional form and modify a constraint, say, the second one, to rule out this solution, but keep all integer solutions as follows.

- Let $[x]$ be the largest integer smaller than $x \in \mathbb{R}$.
- Change the second constraint

  $x_1 + 1.25s_1 + 0.25s_2 = 3.75$ in the final tableau to:

  $x_1 - 2s_1 + 0.75s_1 + 0s_2 + 0.25s_2 = 3 + 0.75$, i.e.,

  $x_1 - 2s_1 + 0s_2 - 3 = 0.75 - 0.75s_1 - 0.25s_2$.
- Now we add the**cut** constraint:

  $0.75 - 0.75s_1 - 0.25s_2 \leq 0$ so that

  1) Any feasible point for the IP will satisfy the cut.

  2) The current optimal solution to the LP relaxation will not satisfy the cut.
- We can now solve the LP problem with the new cut constraint. (See Figure 33 and Table 84.)
- Use dual simplex method to get the solution in Table 85: $z = 40, (x_1, x_2) = (5, 0)$.
- Recall that a cut does not eliminate any points that are feasible for the IP.
- We can apply several cuts until we solve the IP.

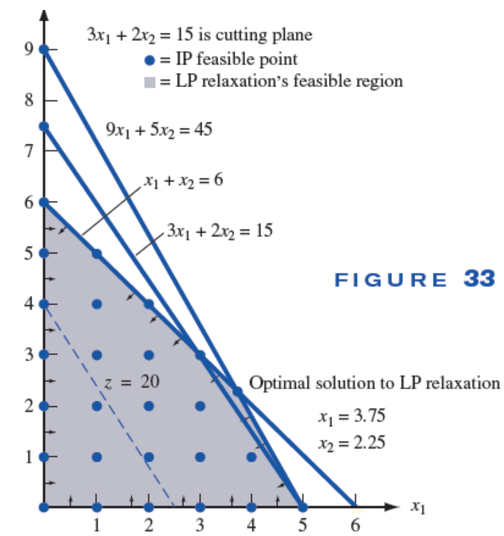


FIGURE 33

TABLE 84
Cutting Plane Tableau After Adding Cut (55)

z	x_1	x_2	s_1	s_2	s_3	rhs
0	0	1	2.25	−0.25	0	2.25
0	1	0	−1.25	0.25	0	3.75
0	0	0	−0.75	−0.25	1	−0.75
1	0	0	1.25	0.75	0	41.25

TABLE 85
Optimal Tableau for Cutting Plane

z	x_1	x_2	s_1	s_2	s_3	rhs
1	0	0	0	0.33	1.67	40
0	0	1	0	−1	3	0
0	1	0	0	0.67	−1.67	5
0	0	0	1	0.33	−1.33	1