Math 323 Operations Research **A research problem.** C.K. Li

Known result (Birkhoff Theorem)

Consider the (balanced) transportation problem: $\min Z = \sum_{i,j=1}^{n} x_{ij}$

subject to $\sum_{j=1}^{n} x_{ij} = 1$ for $i = 1, \ldots, n$

$\sum_{i=1}^{n} x_{ij} = 1$ for $j = 1, \ldots, n$

$x_{ij} \geq 0.$

Problem 1 (10 points homework credit).

(a) Show that for $n = 3$ there are 6 different basic feasible solutions.

(b) Show that for $n = 4$ there are 24 different basic feasible solutions.

Solution. (a) If we write down the constraint equations $Ax = b$, we see that the matrix A has rank 5 because the first five rows are linearly independent, and adding all the rows of A is the zero row. So, we can remove one row to get the reduced system $\tilde{A}\tilde{x} = \tilde{b}$. For a basic feasible solution, there are 5 basic variables that could assume nonzero values. Thus, the matrix (x_{ij}) corresponding to a basic feasible solution has at most 5 nonzero entries. It follows that one of the rows of X has only one nonzero entry, which must equal 1. Then the corresponding column will also has exactly one nonzero entry equal to 1. Let $x_{ij} = 1$. Removing the ith row and the jth column, we get an 2×2 matrix $\tilde{X}$ with row sums and column sums 1 and nonnegative entries. If X is a basic feasible solution, then so must $\tilde{X}$. Otherwise, $\tilde{X} = (\tilde{X}_1 + \tilde{X}_2)/2$ for two different matrices in the feasible solution sets of the 2×2 matrix case. One can extend $\tilde{X}_1, \tilde{X}_2$ to 3×3 matrices X_1, X_2 so that X_1, X_2 have (i, j) entry equal to one, and removing their ith row and jth column yields $\tilde{X}_1, \tilde{X}_2$. Then $X = (X_1 + X_2)/2$ for two different matrices in the feasible set so that X is not a basic feasible solution. Now, every basic feasible solution for the 2×2 matrix case has at most 3 nonzero entries, i.e., at least one zero entry. But then it follows that the matrix has one entry in each row and column.

Thus, we see that if X is a basic feasible solution, then each row and each column has one nonzero entry equal to one. Thus, $x_{1j} = 1$ for some $j \in \{1, 2, 3\}$. Then $x_{2\ell} = 1$ for some $\ell \in \{1, 2, 3\} - \{j\}$. Finally, $x_{3m} = 1$ for $m = \{1, 2, 3\} =\}j, \ell\}$. Thus, there are 6 different matrices that could be basic feasible solutions. In each of this cases, we can go back to $\tilde{A}\tilde{x} = \tilde{b}$ and show that $\tilde{x} = B^{-1}\tilde{b}$ so a suitable choice of B consisting linearly independent columns of $\tilde{A}$. Thus, we get ALL the basic feasible solutions.

(b) Similar to (a).

Problem 2 (10 points for mid-term credit).

Show that every basic feasible solution (x_{ij}) has n nonzero entries all equal to 1 lying in n different rows and columns. (Such a matrix (x_{ij}) is a permutation matrix.)

Solution. We prove the result by induction on n. When $n = 1$, the result is clear. Suppose the result holds for matrices of size less than n. If X is a basic feasible solution, we can write down the constraint equations $Ax = b$ and see that the matrix A has rank 5 because the first five rows are linearly independent, and adding all the rows of A is the zero row. So, we can remove one row to get the reduced system $\tilde{A}\tilde{x} = \tilde{b}$. For a basic feasible solution, there are $2n - 1$ basic variables that could assume nonzero values. Thus, the matrix (x_{ij}) corresponding to a basic feasible solutions must have at most $2n-1$ nonzero entries, and one of the rows of X has only one nonzero entry, which must equal 1. Then the corresponding column will also has exactly one nonzero entry equal to 1. Let $x_{ij} = 1$. Removing the ith row and the jth column, we get an 2×2 matrix $\tilde{X}$ with row sums and column sums 1 and nonnegative entries. If X is a basic feasible solution, then so must $\tilde{X}$. Otherwise, $\tilde{X} = (\tilde{X}_1 + \tilde{X}_2)/2$ for two different matrices in the feasible solution sets of the $(n-1)$-by-$(n-1)$ matrix case. One can extend $\tilde{X}_1, \tilde{X}_2$ to $3ntimesn$ matrices X_1, X_2 so that X_1, X_2 have (i,j) entry equal to one, and removing their ith row and jth column yields $\tilde{X}_1, \tilde{X}_2$. Then $X = (X_1 + X_2)/2$ for two different matrices in the feasible set so that X is not a basic feasible solution. Now, every basic feasible solution for the $(n-1)$-by-$(n-1)$ matrix case must be a permutation matrix. So, X is a permutation matrix. As a result, we see that if X is a basic feasible solution, then each row and each column has one nonzero entry equal to one. Now, to show that such a matrix is a indeed a basic feasible solution, we show that such a matrix X cannot be written as the average of two matrices X_1, X_2 in the feasible solution set. Suppose it does. If $x_{ij} = 0$, then the corresponding entries of X_1, X_2 must also be zero. Else, one of them is positive and the other one is negative. So, X_1, X_2 have the same zero pattern as X. Then the remaining entry in each row and column must be 1. So, $X = X_1 = X_2$, a contradiction. Thus, ALL the basic feasible solutions are permutation matrices.

General (open) problem Consider

Consider the 3-dimensional transportation problem: min $Z = \sum_{i,j,k=1}^{n} x_{ijk}$

subject to $\sum_{k=1}^{n} x_{ijk} = 1$ for all $1 \le i, j \le n$

$\sum_{j=1}^{n} x_{ijk} = 1$ for all $1 \le i, k \le n$

$\sum_{i=1}^{n} x_{ijk} = 1$ for all $1 \le j, k \le n$

$x_{ijk} \ge 0.$

Note There are $3n^2 - 3n + 1$ linearly independent equalities.

For $n = 2$, all basic feasible solutions has entries 1 and 0.

Problem 3 (5 points for final examination credit) For $n = 3$, all basic feasible solutions has entries 1, 0, 1/2. You may list (x_{ijk}) as

$$\left|\begin{array}{ccc|ccc|ccc} x_{111} & x_{112} & x_{113} & x_{211} & x_{212} & x_{213} & x_{311} & x_{312} & x_{313} \\ x_{121} & x_{122} & x_{123} & x_{221} & x_{222} & x_{223} & x_{321} & x_{322} & x_{323} \\ x_{131} & x_{132} & x_{133} & x_{231} & x_{232} & x_{233} & x_{331} & x_{332} & x_{333} \end{array}\right| \text{ to help analyze the problem.}$$

Problem 4 (10 points for final examination credit) Determine all the possible entries of basic feasible solutions for $n = 4$.

Problem 5 (20 points for final examination credit) Determine all the possible entries of basic feasible solutions for $n = 5$.

Remark Note that one need a computer search or delicate argument to show that ALL basic feasible solutions have the said entries.