Solving linear systems

Tablau

- We use matrix notation to record a linear system $Ax = b$, where

  $A = [a_{ij}]$ is $m \times n$, $x = [x_1, \ldots, x_n]^T$ is $n \times 1$ with variables $x_1, \ldots, x_n$, and $b = [b_1, \ldots, b_m]^T$.

- We use Guassian-Jordan reduction on the augmented matrix $[A|b]$ to determine all the solutions.
- In particular, apply elementary row operations and reduce it to the row echelon form.

Examples

$$\left.\begin{aligned} 3x_1 + x_2 + x_3 &= 1 \\ 6x_1 + 2x_2 + 2x_3 &= 2 \end{aligned}\right\} \quad \begin{bmatrix} 3 & 1 & 1 \\ 6 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\left\{ \begin{array}{ccc} x_1 & x_2 & x_3 \\ \end{array} \right.$$

$$\left[\begin{array}{ccc|c} 3 & 1 & 1 & 1 \\ 6 & 2 & 2 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 3 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Row echelon form

Solution:

$$\left\{ \begin{bmatrix} \frac{1}{3} - \frac{1}{3}x_2 - \frac{1}{3}x_3 \\ x_2 \\ x_3 \end{bmatrix} \quad x_2, x_3 \in \mathbb{R} \right\}.$$

- Three possibilities: (a) no solution, (b) one solution, (c) infinitely many solution.

  In terms of the (reduced) row echelon form:

  (a) happens if there is a row of the form $[0, \ldots, 0, r]$ with $r \neq 0$ in the row echelon form;

  (b) happens if (a) does not hold and there are n basic variables (corresponding to leading ones);

  (c) happens if (a) does not hold and there are non-basic variables.

Examples:

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 3 & 6 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 0 & -1 \end{array}\right]$$

The system has no solution

$$\cancel{\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}} \quad \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & -1 & 2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -2 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & -\frac{1}{2} \end{array}\right]$$

$$\to \left[\begin{array}{cc|c} 1 & 0 & \frac{3}{2} \\ 0 & 1 & -\frac{1}{2} \end{array}\right]$$

$$\cancel{\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \end{array}\right]} \quad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & -1 & 2 & 5 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -2 & 1 & 4 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & -\frac{1}{2} & -2 \end{array}\right]$$

Case 1° $\quad x_3 = 0, \quad x_2 = -2, \quad x_1 = 3$

Case 2° $\quad x_2 = 0, \quad x_3 = 4, \quad x_1 = 1 - x_3 = 1 - 4 = -3.$

Case 3° $\quad x_1 = 0, \quad \left[\begin{array}{cc|c} 1 & 1 & 1 \\ -2 & 1 & 4 \end{array}\right] \to \begin{array}{c} \begin{array}{cc} x_2 & x_3 \end{array} \\ \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 3 & 6 \end{array}\right] \end{array}$

$x_3 = 2$

$x_1 = 1 - x_3 = 1 - 2 = -1$