Inverse

- If A is $n \times n$, and there is B such that $AB = I_n$ or $BA = I_n$, then A is invertible, and B is called the **inverse** of A, denoted by A^{-1}. / Inverse may not exist. e.g. $[0]$, $\underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} B = \begin{bmatrix} * & * \\ 0 & 0 \end{bmatrix}}$ $\therefore \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ has no inverse
- If $Ax = b$, where A is $n \times n$ invertible, then $x = A^{-1}b$ is the unique solution of the system.
- One can use the elementary row operations to change $[A|I_n]$ to $[I_n|A^{-1}]$.

Examples

Remark: Even for $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

We want to solve

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Example $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ To find A^{-1}, Consider

$$[A|I_2] = \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & -2 & -3 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & 3/2 & -1/2 \end{array}\right].$$

$$\therefore A^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

Rank, linearly independence

- The **rank** of a matrix A is the number of leading ones in its row echelon form.
- Let A be a matrix. Then A and A^T have the same rank.
- The vectors $\{v_1, \ldots, v_r\}$ in $\mathbb{R}^n$ is **linearly independent** if $A = [v_1|\cdots|v_r]$ has rank r so that the linear system $Ax = 0$ has only the trivial solution $[x_1, \ldots, x_r]^T = [0, \ldots, 0]^T$.

Examples

$$x_1 v_1 + \cdots + x_r v_r = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^n \quad , v_1, \ldots, v_r \in \mathbb{R}^n$$

$$\Leftrightarrow \quad [v_1|\cdots|v_r] \begin{bmatrix} x_1 \\ \vdots \\ x_r \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^n$$

• The **determinant** of an $n \times n$ matrix $A = [a_{ij}]$ is defined by

$$\det(A) = a_{11}\det(A_{11}) - a_{12}\det(A_{12}) + a_{13}\det(A_{13}) - \cdots + (-1)^{n+1}\det(A_{1n}).$$

• An $n \times n$ matrix is invertible if and only if its determinant is nonzero, equivalently, the columns (rows) of A form a linearly independent set.

Examples

$$\det\left(\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}\right)$$

$$= 1\cdot\det\begin{pmatrix}5&6\\8&9\end{pmatrix} - 2\det\begin{pmatrix}4&6\\7&9\end{pmatrix} + 3\det\begin{pmatrix}4&5\\7&8\end{pmatrix}$$

$$= 1.(5\cdot9-6\cdot8) - 2(4\cdot9-7\cdot6) + 3(4\cdot8-5\cdot7)$$

$$= 0$$

Math 323 Operations Research **Notes on Chapter 3.** C.K. Li

Basic format and terminology

- A linear programming (LP) problem is an optimization problem which can be formulated with a **linear objective function** and **linear inequalities and equalities** constraints on some **nonnegative** (decision) variables.

- Specifically, if the (decision) variables are $x_1, \dots, x_n$, then the problem takes the form:

$$\text{maximize/minimzie} \quad Z = c_1x_1 + \cdots + c_nx_n$$

Subject to the constraints

$$\begin{aligned} a_{i1}x_1 + \cdots + a_{in}x_n &\le b_i, && i = 1, \dots, p, \\ a_{i1}x_1 + \cdots + a_{in}x_n &= b_i, && i = p+1, \dots, p+q, \\ x_1, \dots, x_n &\ge 0. \end{aligned}$$

Example

$$3x_1 - x_2 + 5x_3$$

$$x_1 - x_2 + x_3 \le 2$$
$$2x_1 + 5x_2 + 3x_3 \le 1$$
$$x_1 + 2x_2 + x_4 = 1$$
$$x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0$$

Remarks

1. The change of objective function varies as c_ix_i for each $i = 1, \dots, n$, and the variables are independent.

2. The values c_i's, b_j's and a_{ij}'s are all determined with certainly, i.e., not a function of time or subject to noise.

3. To solve the LP problem, we need to determine whether the **feasible region**, i.e., the region containing $(x_1, \dots, x_n)$ satisfying the constraints, is non-empty. / Feasible region may be empty. $x_1 + x_2 \le -1$, $x_1 \ge 0$, $x_2 \ge 0$

4. Then find a point in the feasible region (if non-empty) that produce the **optimal solution** (if exists). Example: $\max Z = x_2$, $x_1 - x_2 \ge 2$, $x_1, x_2 \ge 0$

$$\begin{cases} x_1 + x_2 = 1 \\ 2x_1 + 2x_2 = 4 \end{cases} \quad x_1, x_2 \ge 0$$

5. It is easy to change a minimization problem to a maximization problem, and vice versa

$$\begin{pmatrix} \max Z = c_1x_1 + \cdots + c_nx_n \\ \text{Subj to constraints} \end{pmatrix} \Leftrightarrow \begin{pmatrix} \min -Z = -c_1x_1 - \cdots + (-1)c_nx_n \\ \text{subj to constraints} \end{pmatrix}$$

6. It is easy to change a lower bound constraint $x_i \ge \ell_i$ to a nonnegative constraint.

Change of variables. $y_i = x_i - \ell_i$, $y_i + \ell_i = x_i$

$\max\ c_1(y_1+\ell_1) + \cdots + c_n(y_n+\ell_n) = \sum_{i=1}^{n} c_iy_i + L$

Subj to $\sum a_{ij}(y_j + \ell_j) \le b_i$, $i = 1, \dots, p$

$\sum a_{ij}(y_j + \ell_j) = b_i$, $i = p+1, \dots, p+q$.

7. It is easy to convert one equality constraint to two inequality constraints.

$\max\ Z = y_1 + 2y_2$

Subj to

$y_1 - y_2 \le 0$

$y_1 + 5y_2 \le 8$

~~$y_1 + 5y_2 \ge 8$~~ $-y_1 - 5y_2 \le 8$

$y_1 \ge 0,\ y_2 \ge 0.$

Max $x_1 + 2x_2$

Subj. $x_1 - x_2 \le 2$

$x_1 + 5x_2 = 4$

$x_1 \ge 1,\ x_2 \ge -1.$

Let $y_1 = x_1 - 1$, $y_1 + 1 = x_1$

$y_2 = x_2 + 1$, $y_2 - 1 = x_2$

max $(y_1+1) + 2(y_2-1) = y_1 + 2y_2 - 1$

$(y_1+1) - (y_2-1) \le 2$, i.e., $y_1 - y_2 \le 0$

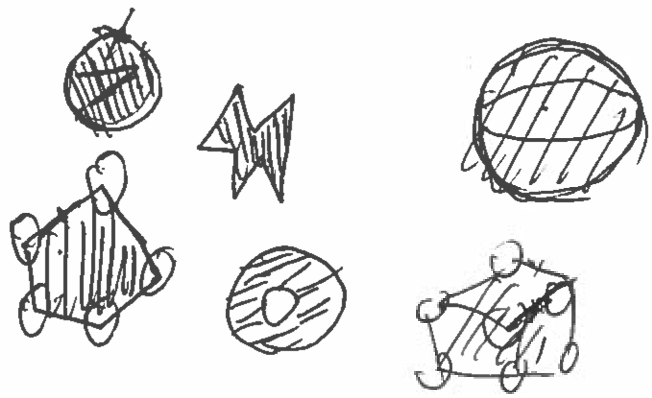
$(y_1+1) + 5(y_2-1) = 4$, i.e., $y_1 + 5y_2 = 8$

3.2 Some terminology associated with convex sets

1. The feasible solution set is a convex set, i.e., a line segment joint two points in the set lie in the set.

2. Examples.

In $\mathbb{R}$ pts in the line joining x_1 & x_2 is the set $\{tx_1+(1-t)x_2 : 0 \le t \le 1\}$



In $\mathbb{R}^2$: Line joining (x_1,y_1), (x_2,y_2)

$\{t(x_1,y_1)+(1-t)(x_2,y_2) : 0 \le t \le 1\}$

3. An extreme point of a convex set is a point that is not the mid-point of two different points in the set. (Any line segment in the set containing the point must have the point as an end points.)

4. The feasible solution set is actually polygonal, i.e., defined by a finite set of linear inequalities.

5. The extreme points of a polygonal set is a vertex or corner point.