3.2 Graphical solution

Example Giapetto's Woodcraving, Inc. produces

- Two toys - soldiers and trains.
- Profit equals selling price minus material cost and production cost (measure in capentry hours).
- For soldiers \$(27-10-14) = \$3; for trains \$(21-9-10) = \$2.
- Every week, no more than 100 unit of material can be used, where every soldier requires 2 unit of material, every train requires 1 unit of material.
- Every week no more than 80 carpentry hours can be used, where where every soldier requires 1 hour, every train requires 1 hour.
- Also, no more than 40 soldiers are needed every week.
- We can set up the LP program by setting x_1, x_2 as the numbers of soldiers and trains made in a week.

$$\max\ Z = 3x_1 + 2x_2$$

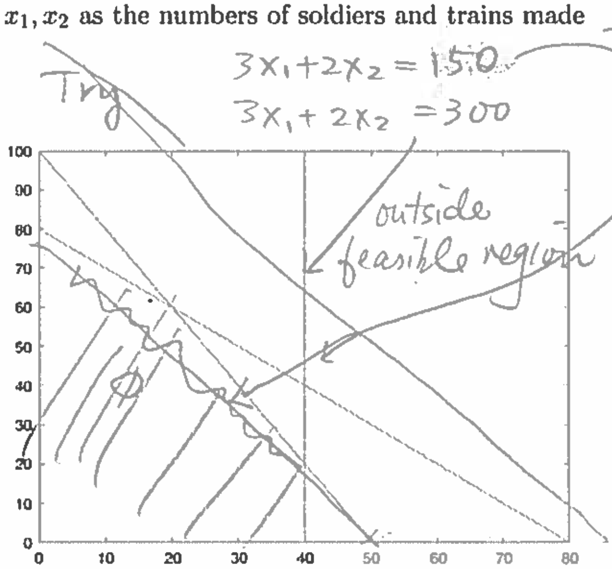
subject to

$$2x_1 + x_2 \leq 100$$

$$x_1 + x_2 \leq 80$$

$$x_1 \quad \leq 40$$

$$x_1, x_2 \geq 0.$$

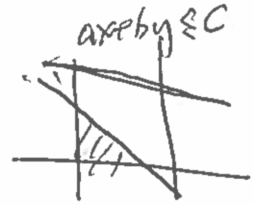


touch many points in the feasible region

1. We consider lines of the form $Z = 3x_1 + 2x_2$ and determine the maximum Z.
2. The (x_1, x_2) values on the same line will yield the same profit Z; so the line is called **isoprofit line**. In other problem, it has other name, say, **isocost line.**
3. For our problem, the optimal value Z occurs at $(x_1, x_2) = (20, 60)$.
4. At the optimal solution, there are binding constraints (equality holds) and unbinding constraints (strict inequality holds).

3.3 Special cases

Remark:



- It may happen that there are multiple optimal solutions.

If $Z = ax_1 + bx_2$ ~~[illegible]~~ and there is a constraint of the form $ax_1 + bx_2 \leq c$ giving "edge" of the feasible region.

- It may happen that there is no feasible solution.

e.g. Max $Z = [illegible]x_1 + [illegible]x_2$

$$\left.\begin{aligned} x_1 + x_2 &\leq 2 \\ x_1 - x_2 &\leq -4 \end{aligned}\right\} \quad 2x_1 \leq -2$$

$x_1, x_2 \geq 0$

- It may happen that there are unbounded solutions.

Max $Z = x_1 + x_2$

$x_1 - x_2 \leq 2$

$x_1, x_2 \geq 0$

Can take $x_2 = M > 0$

$x_1 = M + 2$

Then $Z = 2M + 2$

3.4 – 3.11 Many specific (LP) modeling problems.

Math 323 Operations Research **Notes on Chapter 4.** C.K. Li

The Simplex Algorithms

§4.1 - 4.6 The basic procedures and the theory behind

Standard form

To apply simplex method, we will first change the problem to a standard form:

$$\max Z = c_1x_1 + \cdots + c_m x_n$$

Subject to

$$a_{i1}x_1 + \cdots + a_{in}x_n = b_i, \qquad i = 1, \ldots, m.$$

$$x_1, \ldots, x_n \geq 0.$$

- For

$$\min Z = c_1x_1 + \cdots + c_nx_n$$

we can change it to

$$\max(-Z) = -(c_1x_1 + \cdots + c_nx_n).$$

- For the inequalities

$$c_{i1}x_1 + \cdots + c_{in}x_n \leq b_i, \quad i = 1, \ldots, p,$$

we can add slack variables $s_1, \ldots, s_p \geq 0$ to get

$$c_{i1}x_1 + \cdots + c_{in}x_n + s_i = b_i, \quad i = 1, \ldots, p.$$

- For the inequalities

$$c_{i1}x_1 + \cdots + c_{in}x_n \geq b_i, \quad i = p+1, \ldots, m,$$

$-C_{i1}x_1 - C_{i2}x_2 \cdots - C_{in}x_n \leq -b_i$

$\downarrow$

$-C_{i1}x_1 - C_{i2}x_2 \cdots - C_{in}x_n + e_i = -b$

$\downarrow$

$C_{i1}x_1 + C_{i2}x_2 + \cdots + C_{in}x_n - e_i = b$

we can subtract excess variables $e_{p+1}, \ldots, e_m \geq 0$ to get

$$c_{i1}x_1 + \cdots + c_{in}x_n - e_i = b_i, \quad i = p+1, \ldots, m.$$

Simplex algorithm when $m \leq n$

Step 1. Start with a basic feasible solution (**if it exists**).

Step 2. Improve the basic feasible solution by finding another basic solution (**if possible**) by changing one of the basic variable to go to an adjacent basic solution.

Step 3. Repeat Step 2 until we cannot improve our solution.

Example $\max Z = 5x_1 + 2x_2 + 3x_3 - x_4 + x_5$

Subject to:

$$x_1 + 2x_2 + 2x_3 + x_4 \qquad = 8$$

$$3x_1 + 4x_2 + x_3 + \qquad + x_5 = 7$$

$$x_1, \ldots, x_5 \geq 0.$$

Step 1. Let $(x_4, x_5) = (8, 7)$, $x_1 = x_2 = x_3 = 0$.

Step 2. Let us bring in the nonbasic variable x_1, and consider

$$x_1 + x_4 = 8, \quad 3x_1 + x_5 = 7.$$

Increase $x_1 = 1$ to get the solution $(x_1, x_4, x_5) = (1, 7, 4)$ and $x_2 = x_3 = 0$.
The change in Z value:

$$[5(1) + 2(0) + 3(0) - 1(7) + 1(4)] - [5(0) + 2(0) + 3(0) - 1(8) + 1(7)] = 2 - (-1) = 3,$$

which is an improvement known as the relative profit of the nonbasic variable x_1.
The maximum change limited by the change of x_4, x_5:

$$x_1 + x_4 = 8 \text{ implies } x_1 \leq 8; \quad 3x_1 + x_5 = 7 \text{ implies } x_1 \leq 7/3.$$

So, we may consider the new basic solution $(x_1, x_4) = (7/3, 17/3)$ and $x_2 = x_3 = x_5 = 0$.
The value Z will increase by $(7/3) * 3 = 7$.
Now repeat Step 2 if we can improve.

Step 3. If we cannot improve for any adjacent basic feasible solution, then we get an optimal solution. **(Proof?)**

Simplex methods in Tableau form

C_B	B	$(+5)x_1$	$(+2)x_2$	$(+3)x_3$	$(-1)x_4$	$(+1)x_5$	constraints
-1	x_4	1	2	2	1	0	8
1	x_5	3	4	1	0	1	7

$$Z = (-1, 1)\begin{pmatrix}8\\7\end{pmatrix} = -1.$$

Check the relative profits for different nonbasic variables:

$$\tilde{C}_1 = 5 - (-1, 1)\begin{pmatrix}1\\3\end{pmatrix} = 3, \quad \tilde{C}_2 = 2 - (-1, 1)\begin{pmatrix}2\\4\end{pmatrix} = 0, \quad \tilde{C}_3 = 3 - (-1, 1)\begin{pmatrix}2\\1\end{pmatrix} = 4.$$

We can include the information in the tableau.

C_B	B	$(+5)x_1$	$(+2)x_2$	$(+3)x_3$	$(-1)x_4$	$(+1)x_5$	constraints
-1	x_4	1	2	2	1	0	8
1	x_5	3	4	1	0	1	7
	$\tilde{C}$	3	0	4	0	0	$Z = -1$

Because $\tilde{C}_3$ is largest, we bring in the nonbasic variable x_3.

The limit for x_3 increase is determined by the quotients of the entries in $\begin{pmatrix}x_4\\x_5\end{pmatrix} = \begin{pmatrix}8\\7\end{pmatrix}$ divided by those of the vector under x_3 yielding $\begin{pmatrix}8/2\\7/1\end{pmatrix}$. So, we can increase x_3 by 4, and change x_4 to 0.

We can now update the tableau to by Gaussian elimination:

C_B	B	$(+5)x_1$	$(+2)x_2$	$(+3)x_3$	$(-1)x_4$	$(+1)x_5$	constraints
3	x_3	1/2	1	1	1/2	0	4
1	x_5	5/2	3	0	-1/2	1	3
	$\tilde{C}$	1	-4	0	-2	0	$Z = 15$

So, we can use x_1 as a basic variable. The increase of x_1 is limited by

$$8 \text{ for } x_3, \qquad 6/5 \text{ for } x_5.$$

We can increase $x_1 = 6/5$ and decrease $x_5 = 0$.

Update the tableau to by Gaussian elimination:

C_B	B	$(+5)x_1$	$(+2)x_2$	$(+3)x_3$	$(-1)x_4$	$(+1)x_5$	constraints
3	x_3	0	2/5	1	3/5	-1/5	17/5
5	x_1	1	6/5	0	-1/5	2/5	6/5
	$\tilde{C}$	0	-26/5	0	-9/5	-2/5	$Z = 81/5$

As $\tilde{C}$ are all non-positive, we have an optimal solution.