§4.7-4.12 Summary of Computational Steps if $m \leq n$

Step 1 Set up the problem in the standard form

$$\max Z = c \cdot x = (c_1, \ldots, c_n) \cdot (x_1, \ldots, x_n) = c_1 x_1 + \cdots + c_n x_n$$

Subject to $\quad Ax = b$ for an $m \times n$ matrix $A = (a_{ij})$ and $b \in \mathbb{R}^m$.

$$x_1, \ldots, x_n \geq 0.$$

Step 2 Find a feasible solution using m basic variables, corresponding to the columns of A forming the invertible matrix B. In fact, if we know B, the center part of the updated tableau (without the first and last row, the first and last column) has the form

$$B^{-1}[A|b].$$

Step 3 Compute $\tilde{C}$ vector. Again, it will be of the form $\tilde{C} = c - (c_B)B^{-1}A$.

See the illustrations next page

Step 4 If $\tilde{c}$ is non-positive, then we have the optimal solution.

Otherwise, go to Step 5.

Step 5 Let $\tilde{c}_i$ be the maximum value in $\tilde{c}$ corresponding to the ith nonbasic variable, and let A_i be the ith column of A.

Let $B^{-1}b = \tilde{b} = (\tilde{b}_1, \ldots, \tilde{b}_m)^T$ and $B^{-1}A_i = (\tilde{a}_1, \ldots, \tilde{a}_n)^T$. Then the reduced system with the potential new basic variable has augmented matrix

$$[\tilde{a}|I_m|\tilde{b}]$$

In case all $\tilde{a}_i \leq 0$, then we can increase x_i as much as possible, and we will get an **unbounded** solution.

If $\tilde{a}_j > 0$ for some $j = 1, \ldots, m$, let $\tilde{b}_j/\tilde{a}_j$ be the maximum and replace the the basic variable x_j by the nonbasic variable x_i.

Return to Step 2.

Example (of unbounded solution) If in a maximization problem involving $x_1, \ldots, x_5 \geq 0$ satisfying 2 equations.

Suppose $(x_1, x_2, x_3, x_4, x_5) = (0, 0, 0, 8, 7)$ is a basic feasible solution. If $\tilde{C}_1 = 1 > 0$ and the reduced system is

$$-x_1 + x_4 = 8, \qquad -3x_1 + x_5 = 7.$$

Then we can increase x_1 indefinitely, and conclude that we have an unbounded solution.

Simplex methods in Tableau form

C_B	B	$(+5)x_1$	$(+2)x_2$	$(+3)x_3$	$(-1)x_4$	$(+1)x_5$	constraints
-1	x_4	1	2	2	1	0	8
1	x_5	3	4	1	0	1	7

$[A|b] = [A_1|A_2|A_3|A_4|A_5|b]$

$$Z = (-1, 1)\begin{pmatrix}8\\7\end{pmatrix} = -1.$$

Check the relative profits for different nonbasic variables:

$$\tilde{C}_1 = 5 - (-1, 1)\begin{pmatrix}1\\3\end{pmatrix} = 3, \quad \tilde{C}_2 = 2 - (-1, 1)\begin{pmatrix}2\\4\end{pmatrix} = 0, \quad \tilde{C}_3 = 3 - (-1, 1)\begin{pmatrix}2\\1\end{pmatrix} = 4.$$

We can include the information in the tableau.

C_B	B	$(+5)x_1$	$(+2)x_2$	$(+3)x_3$	$(-1)x_4$	$(+1)x_5$	constraints
-1	x_4	1	2	2	1	0	8
1	x_5	3	4	1	0	1	7
	C	3	0	4	0	0	$Z = -1$

$$\begin{array}{c c} A_4 & A_5 \end{array}$$
$B = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$

$B^{-1}[A|b]$

Because $\tilde{C}_3$ is largest, we bring in the nonbasic variable x_3.

$C - C_B B^{-1}A = (C_1 \cdots C_5) - (-1,1) B^{-1}A$

The limit for x_3 increase is determined by the quotients of the entries in $\begin{pmatrix}x_4\\x_5\end{pmatrix} = \begin{pmatrix}8\\7\end{pmatrix}$ divided by those of the vector under x_3 yielding $\begin{pmatrix}8/2\\7/1\end{pmatrix}$. So, we can increase x_3 by 4, and change x_4 to 0.

We can now update the tableau to by Gaussian elimination:

C_B	B	$(+5)x_1$	$(+2)x_2$	$(+3)x_3$	$(-1)x_4$	$(+1)x_5$	constraints
3	x_3	1/2	1	1	1/2	0	4
1	x_5	5/2	3	0	-1/2	1	3
	C	1	-4	0	-2	0	$Z = 15$

let $B = \begin{bmatrix}2 & 0\\1 & 1\end{bmatrix}$

then

$B^{-1}[A|b]$

So, we can use x_1 as a basic variable. The increase of x_1 is limited by

$$8 \text{ for } x_3, \qquad 6/5 \text{ for } x_5.$$

$C - C_B B^{-1}A$

$= (C_1 \cdots C_5) - (3,1)\begin{bmatrix}2 & 0\\1 & 1\end{bmatrix}^{-1} A,$

We can increase $x_1 = 6/5$ and decrease $x_5 = 0$.

Update the tableau to by Gaussian elimination:

C_B	B	$(+5)x_1$	$(+2)x_2$	$(+3)x_3$	$(-1)x_4$	$(+1)x_5$	constraints
3	x_3	0	2/5	1	3/5	-1/5	17/5
5	x_1	1	6/5	0	-1/5	2/5	6/5
	C	0	-26/5	0	-9/5	-2/5	$Z = 81/5$

let

$B = [A_3|A_1]$

$B^{-1}[A|b]$

As $\tilde{C}$ are all non-positive, we have an optimal solution.

$C - C_B B^{-1}A$

$= (C_1 \cdots C_5) - (3,5)\begin{bmatrix}2 & 1\\1 & 3\end{bmatrix}^{-1} A$