Example (of unbounded solution) If in a maximization problem involving $x_1, \dots, x_5 \geq 0$ satisfying 2 equations.

Suppose $(x_1, x_2, x_3, x_4, x_5) = (0, 0, 0, 8, 7)$ is a basic feasible solution. If $\bar{C}_1 = 1 > 0$ and the reduced system is

$$-x_1 + x_4 = 8, \qquad -3x_1 + x_5 = 7.$$

Then we can increase x_1 indefinitely, and conclude that we have an unbounded solution.

Example of unbounded solution arising in the iteration process

C_B	B	$(2)x_1$	$(+3)x_2$	$(+0)x_3$	$(+0)x_4$	constraints
0	x_3	1	-1	1	0	2
0	x_4	-3	1	0	1	4
	$\bar{C}$	2	3	0	0	$Z = 0$

C_B	B	$(2)x_1$	$(+3)x_2$	$(+0)x_3$	$(+0)x_4$	constraints
0	x_3	-2	0	1	1	6
3	x_2	-3	1	0	1	4
	$\bar{C}$	11	0	0	-3	$Z = 12$

Special cases one may encounter

Alternate Optima

Suppose the iteration leads to:

C_B	B	$(+3)x_1$	$(+2)x_2$	$(+0)x_3$	$(+0)x_4$	$(+0)x_5$	constraints
0	x_3	0	0	1	-1/5	8/5	6
2	x_2	0	1	0	1/5	-3/5	1
3	x_1	1	0	0	1/5	2/5	4
	$\bar{C}$	0	0	0	-1	0	$Z = 14$

The non-basic variable x_5 has zero relative profit. We can use it to replace x_3 and get

C_B	B	$(+3)x_1$	$(+2)x_2$	$(+0)x_3$	$(+0)x_4$	$(+0)x_5$	constraints
0	x_5	0	0	5/8	-1/8	1	15/4
2	x_2	0	1	3/8	1/8	0	13/4
3	x_1	1	0	-1/4	1/4	0	5/2
	$\bar{C}$	0	0	0	-1	0	$Z = 14$

$x = (x_1, x_2, x_3, x_4, x_5) = (4, 1, 6, 0, 0)$, $y = (y_1, y_2, y_3, y_4, y_5) = (\frac{5}{2}, \frac{13}{4}, 0, 0, \frac{15}{4})$

Unique optimum

If all the non-basic variables has negative relative-profit, then the problem has a unique solution.

For $t \in [0,1]$

$tx + (1-t)y$ is an optimal solution.

Ties in the selection of non-basic variable

Suppose $\tilde{c}_i$ is maximum and in the selection of basic variable x_j to be replaced, there are ties in the minimum ratio $\tilde{b}_j/\tilde{a}_j$, then we can choose any one of the x_j to be replaced.

Other degeneracy occurs when a basic variable $x_i = 0$. Then a change of basic variable may lead to no improvement even if we apply the simplex algorithm.

Example

C_B	B	$(0)x_1$	$(+0)x_2$	$(+0)x_3$	$(+2)x_4$	$(+0)x_5$	$(+3/2)x_6$	constraints
0	x_1	1	0	0	1*	-1	0	2
0	x_2	0	1	0	2	0	1	4
0	x_3	0	0	1	1	1	1	3
	$\tilde{C}$	0	0	0	2	0	3/2	$Z = 0$

C_B	B	$(0)x_1$	$(+0)x_2$	$(+0)x_3$	$(+2)x_4$	$(+0)x_5$	$(+3/2)x_6$	constraints
2	x_4	1	0	0	1	-1	0	2
0	x_2	-2	1	0	0	2*	1	0
0	x_3	-1	0	1	0	2	1	1
	$\tilde{C}$	-2	0	0	0	2	3/2	$Z = 4$

C_B	B	$(0)x_1$	$(+0)x_2$	$(+0)x_3$	$(+2)x_4$	$(+0)x_5$	$(+3/2)x_6$	constraints
2	x_4	0	1/2	0	1	0	1/2	2
0	x_5	-1	1/2	0	0	1	1/2	0
0	x_3	1	-1	1	0	0	0	1
	$\tilde{C}$	0	-1	0	0	2	1/2	$Z = 4$

- Two more iterations yield $(x_1, x_4, x_6) = (1, 1, 2)$ as the optimal solution with $Z = 5$.

- In really bad situation, we might even have cycling issue.

- In practice, we are safe if basic feasible solutions always have positive entries (non-degenerate problems).

- As long as there is a positive $\tilde{c}_i$, we should try to improve the solution though it might increase the number of steps in the calculation, but it will not affect the optimal value Z.

§4.12 Finding an initial solution - Big M method, and detecting infeasible problem

Suppose we solve an LP problem

$$\max Z = c_1x_1 + \cdots + c_nx_n$$

subject to

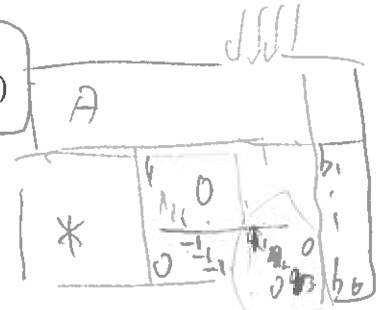
$$\boxed{Ax = b, \quad x_1, \ldots, x_n \geq 0,}$$

where A is $m \times n$ with $m \leq n$.

Changing
$a_{i1}x_1 + \cdots + a_{in}x_n \geq b_i$
to
$a_{i1}x_1 + \cdots + a_{in}x_n - e_i = b_i$

If we have no obvious initial basic feasible solution, we can introduce artificial variable $a_1, \ldots, a_r \geq 0$ and study the problem

$$\max Z = c_1x_1 + \cdots + c_nx_n \boxed{- M(a_1 + \cdots + a_r)}$$



subject to

$$a_{i1}x_1 + \cdots + a_{in}x_n + a_i = b_i, \qquad i \in R,$$

$$a_{i1}x_1 + \cdots + a_{in}x_n = b_j, \qquad i \notin R,$$

$$x_1, \ldots, x_n, a_1, \ldots, a_r \geq 0$$

for a very large $M > 0$, and a suitable subset of $R \subseteq \{1, \ldots, m\}$.

Example $\boxed{\max Z = 2x_1 + 3x_2}$ subject to

$$x_1 + x_2 \leq 8, \quad x_1 + 3x_2 \geq 20, \quad \boxed{x_1, x_2 \geq 0.}$$

$\max Z = 2x_1 + 3x_2 + 0x_3 + 0x_4 - Mx_5$

Then ...

$$x_1 + x_2 + x_3 = 8$$
$$x_1 + 3x_2 - x_4 + x_5 = 20$$
$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

If we cannot get rid to the artificial variable $a_1, \ldots, a_r$ at the end of the process, we have an infeasible problem!

An example of using big M method for a minimization problems

Consider min $Z = -3x_1 + x_2 + x_3 + Mx_6 + Mx_7$

subject to

$$x_1 - 2x_2 + x_3 \leq 11, \quad -4x_1 + x_2 + 2x_3 \geq 3, \quad 2x_1 - x_3 = -1, \quad x_1, x_2, x_3 \geq 0.$$

Adding slack variable $x_4 \geq 0$, excess variable $x_5 \geq 0$ and artificial variables $x_6, x_7 \geq 0$, we get an initial basic feasible solution.

C_B	B	$(-3)x_1$	$(+1)x_2$	$(+1)x_3$	$(+0)x_4$	$(+0)x_5$	$(+M)x_6$	$(+M)x_7$	constraints
0	x_4	1	-2	1	1	0	0	0	11
M	x_6	-4	1	2	0	-1	1	0	3
M	x_7	-2	0	1*	0	0	0	1	1
	C	-3+6M	1-M	1-3M	0	M	0	0	$Z = 4M$

C_B	B	$(-3)x_1$	$(+1)x_2$	$(+1)x_3$	$(+0)x_4$	$(+0)x_5$	$(+M)x_6$	$(+M)x_7$	constraints
0	x_4	3	-2	0	1	0	0	-1	10
M	x_6	0	1*	0	0	-1	1	-2	1
1	x_3	-2	0	1	0	0	0	1	1
	C	-1	1-M	0	0	M	0	3M-1	$Z = M + 1$

C_B	B	$(-3)x_1$	$(+1)x_2$	$(+1)x_3$	$(+0)x_4$	$(+0)x_5$	$(+M)x_6$	$(+M)x_7$	constraints
0	x_4	3*	0	0	1	-2	2	-5	12
1	x_2	0	1	0	0	-1	1	-2	1
1	x_3	-2	0	1	0	0	0	1	1
	C	-1	0	0	0	1	M-1	M+1	$Z = 2$

C_B	B	$(-3)x_1$	$(+1)x_2$	$(+1)x_3$	$(+0)x_4$	$(+0)x_5$	$(+M)x_6$	$(+M)x_7$	constraints
-3	x_1	1	0	0	1/3	-2/3	2/3	-5/3	4
1	x_2	0	1	0	0	-1	1	-2	1
1	x_3	0	0	1	2/3	-4/3	4/3	-7/3	9
	C	0	0	0	1/3	1/3	M-1/3	M-2/3	$Z = -2$

$\min z = x_6 + x_7$

§4.13 The two-phase method

Phase one

C_B	B	$(0)x_1$	$(0)x_2$	$(0)x_3$	$(+0)x_4$	$(+0)x_5$	$(+1)x_6$	$(+1)x_7$	constraints
0	x_4	1	-2	1	1	0	0	0	11
1	x_6	-4	1	2	0	-1	1	0	3
1	x_7	-2	0	1*	0	0	0	1	1
	$\bar{C}$	6	-1	-3	0	1	0	0	$Z = 4$

C_B	B	$(0)x_1$	$(+0)x_2$	$(+0)x_3$	$(+0)x_4$	$(+0)x_5$	$(+1)x_6$	$(+1)x_7$	constraints
0	x_4	3	-2	0	1	0	0	-1	10
1	x_6	0	1*	0	0	-1	1	-2	1
0	x_3	-2	0	1	0	0	0	1	1
	$\bar{C}$	-1	-1	0	0	1	0	3	$Z = 1$

C_B	B	$(0)x_1$	$(+0)x_2$	$(+0)x_3$	$(+0)x_4$	$(+0)x_5$	$(+1)x_6$	$(+1)x_7$	constraints
0	x_4	3	0	0	1	-2	2	-5	12
0	x_2	0	1	0	0	-1	1	-2	1
0	x_3	-2	0	1	0	0	0	1	1
	$\bar{C}$	-1	0	0	0	1	1	1	$Z = 0$

Now move to phase two.

C_B	B	$(-3)x_1$	$(+1)x_2$	$(+1)x_3$	$(+0)x_4$	$(+0)x_5$	constraints
0	x_4	3*	0	0	1	-2	12
1	x_2	0	1	0	0	-1	1
1	x_3	-2	0	1	0	0	1
	$\bar{C}$	-1	0	0	0	1	$Z = 2$

C_B	B	$(-3)x_1$	$(+1)x_2$	$(+1)x_3$	$(+0)x_4$	$(+0)x_5$	constraints
-3	x_1	1	0	0	1/3	-2/3	4
1	x_2	0	1	0	0	-1	1
1	x_3	0	0	1	2/3	-4/3	9
	$\bar{C}$	0	0	0	1/3	1/3	$Z = -2$

$\max z = c_1 x + \cdots + c_n x_n \to (x_n^+ - x_n^-)$

$a_{11}x_1 + \cdots + a_{1n}x_n = b_1$

$\vdots$

$a_{m1}x_1 + \cdots + a_{mn}x_n = b_m$

$x_1, \ldots, x_{n-1} \geq 0$ x_n no sign restriction

let $x_n = x_n^+ - x_n^-$

$x_1, \ldots, x_{n-1}$, $x_n^+, x_n^- \geq 0$

Remarks

- We may also consider variables without sign restriction. See §4.14.
- There are other methods for solving LP: Karmarkar's method, interior point method. For example, see §4.15 and wikipedia.
- One can use build in Matlab commands.

  See https://www.mathworks.com/help/optim/ug/linprog.html

```
To solve minimization problem
                min Z = c_1 x_1 + ... + c_n x_n
  Subject to    Ax \leq b,    AAx = bb,    L \leq x \leq U.
Input c = [c_1, ...., c_n], A, b, AA, bb, L, U.
If no inequality constraints, set A = [], b = [].
Use one of the following commands
    x = linprog(c,A,b)
    x = linprog(c,A,b,AA,bb)
    x = linprog(c,A,b,AA,bb,L,U)
   [x,fval] = linprog(___)
```

Theory behind the simplex algorithm

Theorem 1 *A point in the feasible region of an LP is an extreme point if and only if it is a basic feasible solution of the LP.*

Proof. Every point in $\mathbb{R}^m$ is uniquely determine by m linearly independent equation in $\mathbb{R}^m$. $\square$

Theorem 2 *Suppose an LP in standard form have basic feasible solutions $v_1, \ldots, v_k$. Then every point in the feasible region has the form $v = v_0 + \sum_{j=1}^k p_j v_j$, where v_0 is the zero vector or a vector in the unbounded direction, $p_1, \ldots, p_k$ are non-negative numbers summing up to one.*

Proof. By the theory of convex analysis. $\square$

Theorem 3 If an maximization LP has an optimal solution, then it has an optimal basic feasible solution.

Proof. If an *LP* has an optimal solution Z^* with objective function $\max Z = c \cdot x = c_1 x_1 + \cdots + c_n x_n$, then for any unbounded direction v_0, we have $c \cdot M v_0 \leq Z^*$. So, $c \cdot v_0 = 0$. So, if $v = v_0 + \sum_{j=1}^k p_j v_j$ attains the maximum, we have

$$c.(v_0 + \sum_{j=1}^k p_j v_j) = c \cdot (\sum_{j=1}^k p_j v_j) = \sum_{j=1}^k p_j c \cdot v_j \leq \max\{c \cdot v_j : 1 \leq j \leq k\}. \quad \square$$