Variations of Simplex Algorithm

Goal programming

Suppose the feasible region for a set of constraints is empty. One may want to set up an optimization problem to minimize the damage caused by the artificial variables.

Consider Example 10 in p. 191 in the textbook.

Let x_1 be the number of minutes ad. in football game, and x_2 be the number of minutes of TV ad.

There are three targeted groups, HIM, LIP, HIW, and the Budget constraints:

$$\begin{aligned} 7x_1 + 3x_2 &\geq 40 & \text{(HIM constraint)} \\ 10x_1 + 5x_2 &\geq 60 & \text{(LIP constraint)} \\ 5x_1 + 4x_2 &\geq 35 & \text{(HIW constraint)} \\ 100x_1 + 60x_2 &\leq 600 & \text{(Budget constraint)} \\ x_1, x_2 &\geq 0. & \end{aligned}$$

We can introduce deviational variables s_i^+, s_i^- and solve the following.

$$\min Z = \boxed{200s_1^- + 200s_2^- + 50s_3^-}$$

subject to:

$$\begin{aligned} 7x_1 + 3x_2 + s_1^- - s_1^+ &= 40 & \text{(HIM constraint)} \\ 10x_1 + 5x_2 + s_2^- - s_2^+ &= 60 & \text{(LIP constraint)} \\ 5x_1 + 4x_2 + s_3^- - s_3^+ &= 35 & \text{(HIW constraint)} \\ 100x_1 + 60x_2 &\leq 600 & \text{(Budget constraint)} \\ x_1, x_2, +s_1^-, s_1^+, +s_2^-, s_2^+, +s_3^-, s_3^+ &\geq 0. & \end{aligned}$$

Solving the LP problem, we see that $Z = 250$ with $(x_1, x_2, s_1^+, s_3^-) = (6, 0, 2, 5)$ so that Goal 1 and Goal 2 are satisfied.

If one has to pay for the extra budget we can modify the objective function and the budget constraint to:

Min

$$\cancel{\max} Z = 200s_1^- + 100s_2^- + 50s_3^- + \boxed{s_4}$$

$-s_4$

$$100x_1 + 600x_2 \quad 600. \quad + s_4$$

The solution becomes $Z = 100/3$ $(x_1, x_2, s_1^+, s_4) = (13, 10, 1, 100)/3$; all three goals are met.

One may also impose penalty for the value s_i^+ and add them (with suitable weights) to the objective function.

Preemptive Goad Programming

If one does not know the exact weights of the different goals, one may consider the min $Z = P_1 s_1^- + P_2 s_2^- + P_3 s_3^-$ with $P_1 >>> P_2 >>> P_3$ and add the constraints

$$z_1 - P_1 s_1^- = 0,\ z_2 - P_2 s_2^- = 0 \text{ and } z_3 - P_3 s_3^- = 0.$$

Adding $P1\cdot$ (HIM constraint), $P2\cdot$ (LIP constraint), $P3\cdot$ (HIW constraint) to these constraints, we get

$$z_1 + 7P_1 x_1 + 3P_1 x_2 - P_1 s_1^+ = 40P_1,$$

$$z_2 + 10P_2 x_1 + 5P_2 x_2 - P s_2^+ = 60P_2,$$

$$z_3 + 5P_3 x_1 + 4P_3 x_2 - P_3 s_3^+ = 35P_3.$$

Solving the LP problem, we get $(z_1, z_2, z_3) = (0, 0, 5P_3)$ and $(x_1, x_2, s_1^+, s_3^-) = (6, 0, 2, 5)$ showing that Goal 1 and Goal 2 are met.

One may change the order of P_1, P_2, P_3 and get different solutions.

Scaling of the data

For an LP: max $Z = c \cdot x$ with $c = (c_1, \ldots, c_n)$ and $x = (x_1, \ldots, x_n)$ s.t. $Ax = b$, $x \geq 0$.

One may replace $(c, [A|b])$ by $(\gamma c, D[A|b])$ for some suitable positive constant γ and diagonal matrix D so that the entries in γc and $D(A|b)$ are of comparable magnitudes to avoid unnecessary rounding error.

The optimal solution x will not be changed, and the optimal Z value will be changed to γZ.

Revised Simplex Method

To do the sensitivity analysis in the general setting, it is helpful to understand the revised simplex method and the dual LP/.

Set up the LP problem: max $Z = c \cdot x$ subject to $Ax = b$ and $x \geq 0$, with an initial basic feasible solution, where $c = (c_1, \ldots, c_n)$, $x = (x_1, \cdots, x_n)$, A is $m \times n$.

$A \mid b$

Step 1. Set B be the $m \times m$ with columns from A corresponding to the basic variables.

Step 2. Compute $\tilde{C} = c - c_B B^{-1} A$. (Only compute those $\tilde{c}_j$ corresponding to nonbasic variables.)

If $\tilde{C}$ is non-positive, we are done. Else, go to Step 3.

x_{i_1} ... x_{i_m} $B^{-1}A \mid B^{-1}b$

Step 3. Suppose A_i correspond to the maximum $\tilde{c}_i > 0$.

If $B^{-1}A_i = (\tilde{a}_1, \ldots, \tilde{a}_m)^T$ has only non-positive entries, then the problem is unbounded. $\hat{C} =$
Otherwise, go to Step 4.

$\tilde{C} = \boxed{C} - C_B B^{-1} A$

$1 \times n$ $1 \times m$ $m \times m$ $m \times n$

Step 4. Let $B^{-1}b = (\tilde{b}_1, \ldots, \tilde{b}_m)^T$ and let j be such that $\tilde{b}_j/\tilde{a}_j \leq \tilde{b}_k/\tilde{a}_k$ whenever $\tilde{a}_k > 0$.
Replace the basic variable x_j by x_i. Go back to Step 2.
Here we may update the matrix B^{-1} and $B^{-1}b$ for future use:

$1 \times n$

$$B^{-1}[A_i|A_{j1} \cdots A_{j_m}|I_m|b] = [B^{-1}A_i|I_m|B^{-1}|B^{-1}b]$$

$$\rightarrow [e_j|e_1 \ldots e_{j-1}\hat{A}_j e_{j+1} \cdots e_n|\hat{B}^{-1}|\hat{B}^{-1}b].$$

Advantages of the revised simplex method

1. No need to update the whole tableau if there are many variables (columns).

2. No need to store all the information; just the original A, the basic variables, B^{-1}, and $B^{-1}b$.

3. Less error in the iteration because the original A is used in each step.

4. The method is efficient if A is sparse or having other structure.

5. The idea is useful for duality and sensitivity analysis.