Revised Simplex Method

To do the sensitivity analysis in the general setting, it is helpful to understand the revised simplex method and the dual LP/.

Set up the LP problem: $\max Z = c \cdot x$ subject to $Ax = b$ and $x \geq 0$,
with an initial basic feasible solution, where $c = (c_1, \ldots, c_n), x = (x_1, \cdots, x_n)$, A is $m \times n$.

Step 1. Set B be the $m \times m$ with columns from A corresponding to the basic variables.

Step 2. Compute $\tilde{C} = c - c_B B^{-1} A$. (Only compute those $\tilde{c}_j$ corresponding to nonbasic variables.)

If $\tilde{C}$ is non-positive, we are done. Else, go to Step 3.

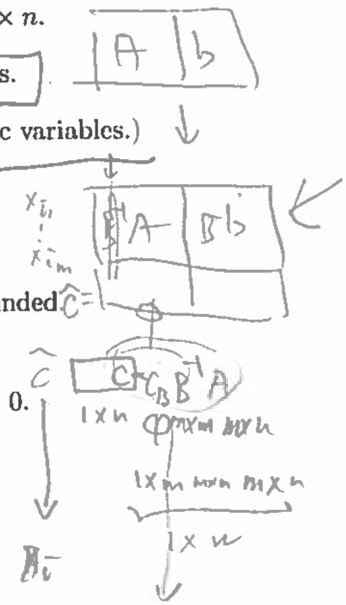
Step 3. Suppose A_i correspond to the maximum $\tilde{c}_i > 0$.

If $B^{-1} A_i = (\tilde{a}_1, \ldots, \tilde{a}_m)^T$ has only non-positive entries, then the problem is unbounded.
Otherwise, go to Step 4.

Step 4. Let $B^{-1} b = (\tilde{b}_1, \ldots, \tilde{b}_m)^T$ and let j be such that $\tilde{b}_j / \tilde{a}_j \leq \tilde{b}_k / \tilde{a}_k$ whenever $\tilde{a}_k > 0$.
Replace the basic variable x_j by x_i. Go back to Step 2.
Here we may update the matrix B^{-1} and $B^{-1} b$ for future use:

$$B^{-1}[A_i | A_{j1} \cdots A_{j_m} | I_m | b] = [B^{-1} A_i | I_m | B^{-1} | B^{-1} b]$$

$$\to [e_j | e_1 \ldots e_{j-1} \hat{A}_j e_{j+1} \cdots e_n | \hat{B}^{-1} | \hat{B}^{-1} b].$$



Advantages of the revised simplex method

1. No need to update the whole tableau if there are many variables (columns).

2. No need to store all the information; just the original A, the basic variables, B^{-1}, and $B^{-1} b$.

3. Less error in the iteration because the original A is used in each step.

4. The method is efficient if A is sparse or having other structure.

5. The idea is useful for duality and sensitivity analysis.

Back to sensitivity analysis

Example A toy company producing three products: x_1, x_2, x_3 and set up the LP

$$\max Z = 2x_1 + 3x_2 + x_3$$

max
$Z = C \cdot x$

Subject to:

$$\frac{1}{3}x_1 + \frac{1}{3}x_2 + \frac{1}{3}x_3 \leq 1 \qquad \text{(labor)}$$

$$\frac{1}{3}x_1 + \frac{4}{3}x_2 + \frac{7}{3}x_3 \leq 1 \qquad \text{(material)}$$

$$x_1, x_2, x_3 \geq 0.$$

$Ax \leq b$

$x \geq 0$

Use simplex algorithm with slack variables $x_4, x_5 \geq 0$:

C_B	B	$(+2)x_1$	$(+3)x_2$	$(+1)x_3$	$(0)x_4$	$(+1)x_5$	constraints
0	x_4	1/3	1/3	1/3	1	0	1
0	x_5	1/3	4/3	7/3	0	1	3
	C	2	3	1	0	0	$Z = 0$

	C_B	B	$(+2)x_1$	$(+3)x_2$	$(+1)x_3$	$(0)x_4$	$(+1)x_5$	constraints
$\to$	2	x_1	1	0	-1	4	-1	1
	3	x_2	0	1	2	-1	1	2
		C	0	0	-3	-5	-1	$Z = 8$

Case 1. Changing c_i corresponding to non-basic variables.

$(c_1 \cdots c_5) - (c_1 c_2) B^{-1} A$

Note that

$$\tilde{c}_3 = c_3 - (2,3) \cdot (-1,2) = c_3 - 4.$$

So, if $c_3 < 4$ we have the same optimum; if $c_3 = 4$ then there may be alternate optimum; if $c_3 > 4$ then there may be improvement.

For instance, in our example, if c_3 is changed to 6, then

C_B	B	$(+2)x_1$	$(+3)x_2$	$(+\cancel{1})x_3$ 6	$(0)x_4$	$(+1)x_5$	constraints
2	x_1	1	0	-1	4	-1	1
3	x_2	0	1	2	-1	1	2
	C	0	0	2	-5	-1	$Z = 8$

	C_B	B	$(+2)x_1$	$(+3)x_2$	$(+\cancel{1})x_3$ 6	$(0)x_4$	$(+1)x_5$	constraints
$\to$	2	x_1	1	1/2	0	7/2	$-1/2$	2
	6	x_3	0	1/2	1	$-1/2$	1/2	1
		C	0	-1	0	-4	-2	$Z = 10$

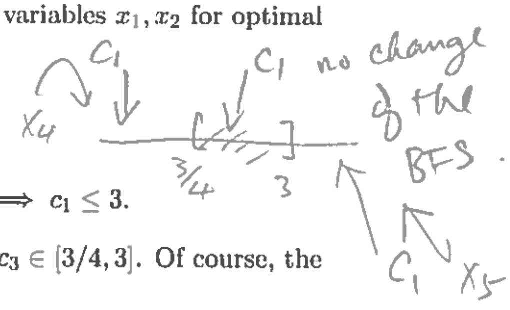
In general, we analyze the change of $\bar{c}_i = c_i - c_B B^{-1} A_i$.

Case 2. Changing c_j corresponding to basic variables.

If we change c_j corresponding to a basic variable x_j, then $\tilde{c} = c - c_B B^{-1} A$ in terms of c_j.

For example, if we consider a variation of c_1 after getting the basic variables x_1, x_2 for optimal in our example, then

$$(\tilde{c}_3, \tilde{c}_4, \tilde{c}_5) = (c_1 - 5, -4c_1 + 3, c_1 - 3).$$



Hence

$$\tilde{c}_3 \le 0 \iff c_1 \le 5 \quad \tilde{c}_4 \le 0 \iff c_1 \ge 3/4, \quad \tilde{c}_5 \le 0 \iff c_1 \le 3.$$

Thus, we will use the same optimal solution $(x_1, \dots, x_5)$ if and only if $c_3 \in [3/4, 3]$. Of course, the optimal value Z will change.

The optimal solution will change otherwise. For example if $c_1 = 1$, then the optimal solution is $(x_1, x_2, x_3, x_4, x_5) = (1, 2, 0, 0, 0)$ with $Z = 7$.

If c_1 goes outside the range, we have to change $\tilde{c}$ and apply the Simplex algorithm again.

In general, we compute the entries in $\tilde{c} = c - c_B B^{-1} A$ corresponding to the non-basic variable to determine the range of change of c_j and action needed.

Case 3 Changing c in general.

If we get a solution for the basic variables x_1, x_3 and we want to change c, then we simply compute

$$\tilde{c} = c - c_B B^{-1} A = (c_1, c_2, c_3, c_4, c_5) - (c_1, c_3) \begin{pmatrix} 1 & 1/2 & 0 & 7/2 & -1/2 \\ 0 & 1/2 & 1 & -1/2 & 1/2 \end{pmatrix}$$

$$= (0, 2c_2 - c_1 - c_2, 0, 2c_4 - 7c_1 + c_3, 2c_5 + c_1 - c_3)/2.$$

and determine the course of action.

Changing the vector b

1. Note that in our example, the final tableau, the last column is $B^{-1}b$ with $B = \begin{pmatrix} 1/3 & 1/3 \\ 1/3 & 4/3 \end{pmatrix}$ so that $B^{-1} = \begin{pmatrix} 4 & -1 \\ -1 & 4 \end{pmatrix}$.

2. Now, if we change b from $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ to $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$, then the last column of the final tableau will change to

$$B^{-1}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}.$$

$B^{-1}A$ $C - C_B B^{-1}A$ $\rightarrow B^{-1}b$

3. Because $B^{-1}A, \tilde{C} = c - c_B B^{-1}A$ do not change, we still have the same basic variables for the solution.

4. But (x_1, x_2, x_3) and Z change to $(5, 1, 0)$ and $Z = 13$, respectively.

5. If increasing a unit of b_1 cost \$4 (overtime cost), and the profit will increase by \$$(13 - 8) =$ \$5. So, it is worth doing the overtime.

6. We will call the profit change corresponding to a unit change of the b_i the **shadow price**.

7. Knowing the shadow price will tell us whether it is worthwhile to increase b_i.

8. In the final tableau, the shadow price corresponding to b_i is computed by

$$c_B B^{-1}(b + e_i) - c_B B^{-1}b = c_B B^{-1}e_i.$$

Thus, the entries in the row $C_B B^{-1}$ tell us the shadow price for each of the m basic variables.

9. In our example, if we let $b^* = \begin{pmatrix} b_1 \\ 3 \end{pmatrix}$, then in the final tableau the last column becomes $B^{-1}b^* = \begin{pmatrix} 4b_1 - 3 \\ -b_1 + 3 \end{pmatrix}$.

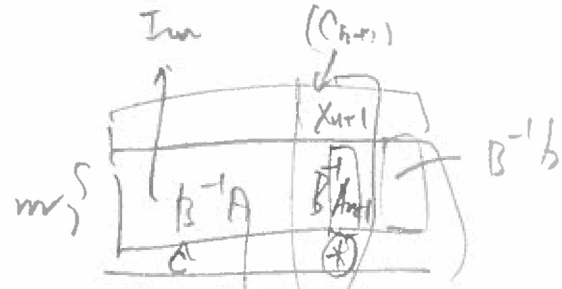
10. So, x_1, x_2 are the basic variables for the optimal solution if $3/4 \le b_1 \le 3$.

11. The optimal value will be $Z = 2(4b_1 - 3) + 3(-b_1 + 3) = 5b_1 + 3$.

12. What if b is changed, say, to $(4, 3)^T$, so that $B^{-1}b$ is no longer feasible? Then the tableau changes to:

C_B	B	$(+2)x_1$	$(+3)x_2$	$(+1)x_3$	$(0)x_4$	$(+1)x_5$	constraints
2	x_1	1	0	-1	4	-1	13
3	x_2	0	1	2	-1^*	1	-1
	C	0	0	-3	-5	-1	$Z = 8$

We will tackle this with dual LP theory.



Changing the matrix A.

Case 1. Adding a new decision variable x_{n+1}.

In our example if we add another product, say, x_6 with a unit profit \$3, and costing 1 unit of labor and 1 unit of material. Then we update the c vector by adding $c_6 = 3$, and add the column $A_6 = (1,1)^T$ in A corresponding to x_6 and compute

$$\tilde{c}_6 = c_6 - c_B B^{-1} A_6 = 3 - (2,3)B^{-1}A_6 = 3 - (5,1)\cdot(1,1) = -3.$$

Because $\tilde{c}_6 \le 0$, we still have the same optimal solution. If $\tilde{c}_6 > 0$, then we run the simplex algorithm.

Case 2. Changing the resources requirements.

We need to change A and ~~B accordingly~~. The solution may no longer be feasible (even if we use the dual LP), and we may need to start all over again.

$A_{:j}$

Case 3. Adding new constraints.

If a new constraint is added, say, $x_1 + 2x_2 + x_3 \le 10$ is the limit of administrative hours.

Check whether the current optimal solution satisfies the constraint. If yes, it will remains to be an optimal solution.

If not, add a slack variable x_6 and consider the modified tableau:

C_B	B	$(+2)x_1$	$(+3)x_2$	$(+1)x_3$	$(+0)x_4$	$(+1)x_5$	$(+0)x_6$	constraints
2	x_1	1	0	-1	4	-1	0	1
3	x_2	0	1	2	-1	1	0	2
0	x_6	1	2	1	0	0	1	4
	$\tilde{C}$	0	0	-3	-5	-1	0	

	C_B	B	$(+2)x_1$	$(+3)x_2$	$(+1)x_3$	$(+0)x_4$	$(+1)x_5$	$(+0)x_6$	constraints
	2	x_1	1	0	-1	4	-1	0	1
$\rightarrow$	3	x_2	0	1	2	-1	1	0	2
	0	x_6	0	0	-2	-2	-1^*	1	-1
		$\tilde{C}$	0	0	-3	-5	-1	0	

where the dual LP theory can be used.

$$m\ (a_{ij})\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \leq \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

Dual LP

Consider the following primal LP: $c_1x_1 + \cdots + c_nx_n$

$$\max Z = \boxed{(c_1, \ldots, c_n) \cdot (x_1, \ldots, x_n)} \quad \text{subject to} \quad Ax \leq (b_1, \ldots, b_m)^T, \quad x_1, \ldots, x_n \geq 0,$$

where $x = (x_1, \ldots, x_n)^T$ and $A = (a_{ij})$ is $m \times n$.

Then the dual LP is defined as $\quad n\times m \quad A^T\begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} \geq \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$

$$\min W = (b_1, \ldots, b_m) \cdot (y_1, \ldots, y_m) \quad \text{subject to} \quad A^T y \geq (c_1, \ldots, c_n)^T, \quad y_1, \ldots, y_m \geq 0,$$

where $y = (y_1, \ldots, y_m)^T$.

Example (Dekota problem, p. 296) Primal problem

$$\max Z = 60x_1 + 30x_2 + 20x_3$$

$$\begin{aligned}
\text{subject to:} \quad & 8x_1 + 6x_2 + x_3 \leq 48 && \text{(Lumber constraint)} \\
& 4x_1 + 2x_2 + 1.5x_3 \leq 20 && \text{(Finishing constraint)} \\
& 2x_1 + 1.5x_2 + 0.5x_3 \leq 8 && \text{(Capentry constriant)} \\
& x_1, x_2, x_3 \geq 0
\end{aligned}$$

Dual problem.

$$\min W = 48y_1 + 20y_2 + 8y_3$$

$$\begin{aligned}
\text{subject to:} \quad & 8y_1 + 4y_2 + 2y_3 \geq 60 \\
& 6y_1 + 2y_2 + 1.5y_3 \geq 30 \\
& y_1 + 1.5y_2 + 0.5y_3 \geq 20 \\
& y_1, y_2, y_3 \geq 0.
\end{aligned}$$

Example (Diet problem)

$$\min W = 50y_1 + 20y_2 + 30y_3 + 80y_4$$

$$\begin{aligned}
\text{subject to:} \quad & 400y_1 + 200y_2 + 150y_3 + 500y_4 \geq 500 && \text{(Calorie constraint)} \\
& 3y_1 + 2y_2 \geq 6 && \text{(Chocolate constraint)} \\
& 2y_1 + 2y_2 + 4y_3 + 4y_4 \geq 10 && \text{(Sugar constraint)} \\
& 2y_1 + 4y_2 + y_3 + 5y_4 \geq 8 && \text{(Fat constraint)} \\
& y_1, y_2, y_3, y_4 \geq 0.
\end{aligned}$$

The Primal problem:

$$\max Z = 500x_1 + 6x_2 + 10x_3 + 8x_4$$

$$\begin{aligned}
\text{subject to:} \quad & 400x_1 + 3x_2 + 2x_3 + 2x_4 \leq 50 \\
& 200x_1 + 2x_2 + 2x_3 + 4x_4 \leq 20 \\
& 150x_1 + 4x_3 + x_4 \leq 30 \\
& 500x_1 + 4x_3 + 5x_4 \leq 80 \\
& x_1, x_2, x_3, x_4 \geq 0.
\end{aligned}$$