Dual LP

Consider the following primal LP:

$$\max Z = (c_1, \dots, c_n) \cdot (x_1, \dots, x_n) \quad \text{subject to } Ax \le (b_1, \dots, b_m)^T, \quad x_1, \dots, x_n \ge 0,$$

where $x = (x_1, \dots, x_n)^T$ and $A = (a_{ij})$ is $m \times n$.

Then the dual LP is defined as

$$\min W = (b_1, \dots, b_m) \cdot (y_1, \dots, y_m) \quad \text{subject to } A^T y \ge (c_1, \dots, c_n)^T, \quad y_1, \dots, y_m \ge 0,$$

where $y = (y_1, \dots, y_m)^T$.

Example (Dekota problem, p. 296) Primal problem

$$\max Z = 60x_1 + 30x_2 + 20x_3$$

$$\begin{array}{lrl}
\text{subject to:} & 8x_1 + 6x_2 + x_3 \le 48 & \text{(Lumber constraint)} \\
& 4x_1 + 2x_2 + 1.5x_3 \le 20 & \text{(Finishing constraint)} \\
& 2x_1 + 1.5x_2 + 0.5x_3 \le 8 & \text{(Capentry constriant)} \\
& x_1, x_2, x_3 \ge 0. &
\end{array}$$

Dual problem.

$$\min W = 48y_1 + 20y_2 + 8y_3$$

$$\begin{array}{lr}
\text{subject to:} & 8y_1 + 4y_2 + 2y_3 \ge 60 \\
& 6y_1 + 2y_2 + 1.5y_3 \ge 30 \\
& y_1 + 1.5y_2 + 0.5y_3 \ge 20 \\
& y_1, y_2, y_3 \ge 0.
\end{array}$$

Example (Diet problem)

$$\min W = 50y_1 + 20y_2 + 30y_3 + 80y_4$$

$$\begin{array}{llrl}
\text{subject to:} & x_1 & 400y_1 + 200y_2 + 150y_3 + 500y_4 \ge 500 & \text{(Calorie constraint)} \\
& x_2 & 3y_1 + 2y_2 \ge 6 & \text{(Chocolate constraint)} \\
& x_3 & 2y_1 + 2y_2 + 4y_3 + 4y_4 \ge 10 & \text{(Sugar constraint)} \\
& x_4 & 2y_1 + 4y_2 + y_3 + 5y_4 \ge 8 & \text{(Fat constraint)} \\
& & y_1, y_2, y_3, y_4 \ge 0. &
\end{array}$$

The Primal problem:

$$\max Z = 500x_1 + 6x_2 + 10x_3 + 8x_4$$

$$\begin{array}{lr}
\text{subject to:} & 400x_1 + 3x_2 + 2x_3 + 2x_4 \le 50 \\
& 200x_1 + 2x_2 + 2x_3 + 4x_4 \le 20 \\
& 150x_1 + 4x_3 + x_4 \le 30 \\
& 500x_1 + 4x_3 + 5x_4 \le 80 \\
& x_1, x_2, x_3, x_4 \ge 0.
\end{array}$$

Remark An interpretation of the dual problem.

Finding the dual LP not in standard primal form
Example

$$\max Z = 2x_1 + x_2$$

subject to

$$x_1 + x_2 = 2$$
$$2x_1 - x_2 \geq 3$$
$$x_1 - x_2 \leq 1$$
$$x_1 \geq 0\ ,\ x_2 \text{ urs.}$$

First set $x_2 = x_2^+ - x_2^-$ with $x_2^+, x_2^- \geq 0$, and convert the problem to

$$\max Z = 2x_1 + x_2^+ - x_2^-$$

subject to

$$x_1 + x_2^+ - x_2^- \leq 2$$
$$-x_1 - x_2^+ + x_2^- \leq -2$$
$$-2x_1 + x_2^+ - x_2^- \leq 3$$
$$x_1 - x_2^+ + x_2^- \leq 1$$
$$x_1, x_2^+, x_2^- \geq 0.$$

The dual LP becomes

$$\min W = 2y_1' - 2y_1'' + 3y_2 + 1y_3$$

subject to

$$y_1' - y_1'' + 2y_2 + y_3 \geq 2$$
$$y_1' - y_1'' + y_2 - y_3 \geq 1$$
$$-y_1' + y_1'' - y_2 + y_3 \geq -1$$
$$y_1', y_1'', y_2, y_3 \geq 0.$$

We set $y_1 = y_1' - y_1''$ and get

The dual LP becomes:

$$\min W = 2y_1 + 3y_2 + y_3$$

subject to

$$y_1 + 2y_2 + y_3 \geq 2$$
$$y_1 + y_2 - y_3 \geq 1$$
$$-y_1 - y_2 + y_3 \geq -1$$
$$y_1 \text{ urs},\ y_2, y_3 \geq 0.$$

Setting $y_1 = y_1' - y_1''$ and $\hat{y}_2 = -y_2$, we get

$$\min W = 2y_1 + 3\hat{y}_2 + y_3$$

subject to

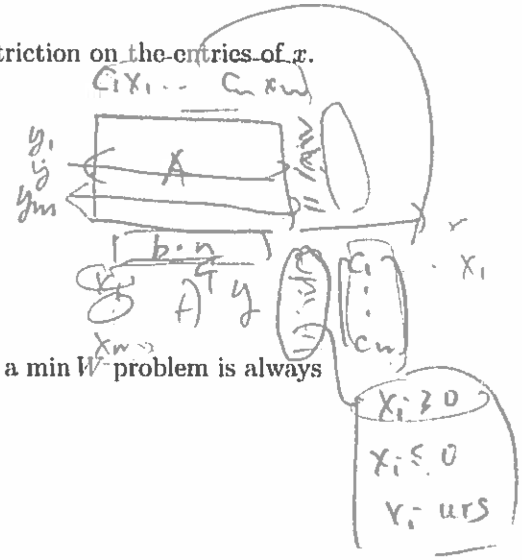
$$y_1 + 2\hat{y}_2 + y_3 \geq 2$$
$$y_1 - \hat{y}_2 - y_3 = 1$$
$$y_1 \text{ urs},\ \hat{y}_2 \leq 0,\ y_3 \geq 0.$$

General rules Consider

$\max Z = c \cdot x$ subject to $Ax = \mathbf{b}_1, Ax \le \mathbf{b}_2, Ax \ge \mathbf{b}_3$. with sign restriction on the entries of x.

The dual becomes: $\min W = b \cdot y$ such that

- If ith constraint is $\le$ type $y_i \ge 0$.
- If ith constraint is $\ge$ type $y_i \le 0$.
- If x_i is unrestricted in sign, then the jth constraint is an equation.
- If $x_i \ge 0$, then then the jth constraint is of the $\ge$ type.
- If $x_i \le 0$, then then the jth constraint is of the $\le$ type.



Remark In fact, the dual of $\max Z$ is always converted to $\min W$, and a $\min W$ problem is always converted to a $\max Z$ problem.

Example 1 Primal LP

$$
\begin{aligned}
& \max Z = x_1 + 4x_2 + 3x_3 \\
\text{subject to} \quad & 2x_1 + 3x_2 - 5x_3 \le 2 \\
& 3x_1 - x_2 + 6x_3 \ge 1 \\
& x_1 + x_2 + x_3 = 4 \\
& x_1 \ge 0, \;\; x_2 \le 0, \;\; x_3 \text{ urs.}
\end{aligned}
$$

Duel LP

$$
\begin{aligned}
& \min W = 2y_1 + y_2 + 4y_3 \\
\text{subject to} \quad & 2y_1 + 3y_2 + y_3 \ge 1 \\
& 3y_1 - y_2 + y_3 \le 4 \\
& -5y_1 + 6y_2 + y_3 = 3 \\
& y_1 \ge 0, \;\; y_2 \le 0, \;\; y_3 \text{ urs.}
\end{aligned}
$$

Example 2 Primal LP

$$
\begin{aligned}
& \min Z = 2x_1 + x_2 - x_3 \\
\text{subject to} \quad & x_1 + x_2 - x_3 = 1 \\
& x_1 - x_2 + x_3 \ge 2 \\
& x_2 + x_3 \le 3 \\
& x_1 \ge 0, \;\; x_2 \le 0, \;\; x_3 \text{ urs.}
\end{aligned}
$$

Duel LP

$$
\begin{aligned}
& \max W = y_1 + 2y_2 + 3y_3 \\
\text{subject to} \quad & y_1 + y_2 \qquad \le 2 \\
& 3y_1 - y_2 + y_3 \ge 1 \\
& -y_1 + y_2 + y_3 = -1 \\
& y_1 \text{ urs}, \;\; y_2 \ge 0, \;\; y_3 \le 0.
\end{aligned}
$$

Remark *The dual of the dual of an LP is the original problem.*