Finding the dual LP not in standard primal form

Example

$$
\begin{aligned}
& \max Z = 2x_1 + x_2 \\
\text{subject to} \quad & x_1 + x_2 = 2 \\
& 2x_1 - x_2 \geq 3 \\
& x_1 - x_2 \leq 1 \\
& x_1 \geq 0\ ,\ x_2 \text{ urs.}
\end{aligned}
$$

First set $x_2 = x_2^+ - x_2^-$ with $x_2^+, x_2^- \geq 0$, and convert the problem to

$$
\begin{aligned}
& \max Z = 2x_1 + x_2^+ - x_2^- \\
\text{subject to} \quad & x_1 + x_2^+ - x_2^- \leq 2 \\
& -x_1 - x_2^+ + x_2^- \leq -2 \\
& -2x_1 + x_2^+ - x_2^- \leq 3 \\
& x_1 - x_2^+ + x_2^- \leq 1 \\
& x_1, x_2^+, x_2^- \geq 0.
\end{aligned}
$$

The dual LP becomes

$$
\begin{aligned}
& \min W = 2y_1' - 2y_1'' + 3y_2 + 1y_3 \\
\text{subject to} \quad & y_1' - y_1'' + 2y_2 + y_3 \geq 2 \\
& y_1' - y_1'' + y_2 - y_3 \geq 1 \\
& -y_1' + y_2'' - y_2 + y_3 \geq -1 \\
& y_1', y_2'', y_2, y_3 \geq 0.
\end{aligned}
$$

We set $y_1 = y_1' - y_1''$ and get

The dual LP becomes:

$$
\begin{aligned}
& \min W = 2y_1 + 3y_2 + y_3 \\
\text{subject to} \quad & y_1 + 2y_2 + y_3 \geq 2 \\
& y_1 + y_2 - y_3 \geq 1 \\
& -y_1 - y_2 + y_3 \geq -1 \\
& y_1 \text{ urs},\ y_2, y_3 \geq 0.
\end{aligned}
$$

Setting $y_1 = y_1' - y_1''$ and $\hat{y}_2 = -y_2$, we get

$$
\begin{aligned}
& \min W = 2y_1 + 3\hat{y}_2 + y_3 \\
\text{subject to} \quad & y_1 + 2\hat{y}_2 + y_3 \geq 2 \\
& y_1 - \hat{y}_2 - y_3 = 1 \\
& y_1 \text{ urs},\ \hat{y}_2 \leq 0,\ y_3 \geq 0.
\end{aligned}
$$

$$n$$
$$m \; (a_{ij}) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \leq \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

Dual LP

Consider the following primal LP: $c_1x_1 + \cdots + c_nx_n$

$$\max Z = \boxed{(c_1, \ldots, c_n) \cdot (x_1, \ldots, x_n)} \quad \text{subject to} \quad Ax \leq (b_1, \ldots, b_m)^T, \quad x_1, \ldots, x_n \geq 0,$$

where $x = (x_1, \ldots, x_n)^T$ and $A = (a_{ij})$ is $m \times n$.

Then the dual LP is defined as $\quad n \times m \quad A^T \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} \geq \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$

$$\min W = (b_1, \ldots, b_m) \cdot (y_1, \ldots, y_m) \quad \text{subject to} \quad A^T y \geq (c_1, \ldots, c_n)^T, \quad y_1, \ldots, y_m \geq 0,$$

where $y = (y_1, \ldots, y_m)^T$.

Example (Dekota problem, p. 296) Primal problem

$$\max Z = 60x_1 + 30x_2 + 20x_3$$

subject to:
$$\begin{aligned} 8x_1 + 6x_2 + x_3 &\leq 48 && \text{(Lumber constraint)} \\ 4x_1 + 2x_2 + 1.5x_3 &\leq 20 && \text{(Finishing constraint)} \\ 2x_1 + 1.5x_2 + 0.5x_3 &\leq 8 && \text{(Capentry constriant)} \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

Dual problem.

$$\min W = 48y_1 + 20y_2 + 8y_3$$

subject to:
$$\begin{aligned} 8y_1 + 4y_2 + 2y_3 &\geq 60 \\ 6y_1 + 2y_2 + 1.5y_3 &\geq 30 \\ y_1 + 1.5y_2 + 0.5y_3 &\geq 20 \\ y_1, y_2, y_3 &\geq 0. \end{aligned}$$

Example (Diet problem)

$$\min W = 50y_1 + 20y_2 + 30y_3 + 80y_4$$

subject to:
$$\begin{aligned} 400y_1 + 200y_2 + 150y_3 + 500y_4 &\geq 500 && \text{(Calorie constraint)} \\ 3y_1 + 2y_2 &\geq 6 && \text{(Chocolate constraint)} \\ 2y_1 + 2y_2 + 4y_3 + 4y_4 &\geq 10 && \text{(Sugar constraint)} \\ 2y_1 + 4y_2 + y_3 + 5y_4 &\geq 8 && \text{(Fat constraint)} \\ y_1, y_2, y_3, y_4 &\geq 0. \end{aligned}$$

The Primal problem:

$$\max Z = 500x_1 + 6x_2 + 10x_3 + 8x_4$$

subject to:
$$\begin{aligned} 400x_1 + 3x_2 + 2x_3 + 2x_4 &\leq 50 \\ 200x_1 + 2x_2 + 2x_3 + 4x_4 &\leq 20 \\ 150x_1 + 4x_3 + x_4 &\leq 30 \\ 500x_1 + 4x_3 + 5x_4 &\leq 80 \\ x_1, x_2, x_3, x_4 &\geq 0. \end{aligned}$$

General rules for converting an LP to its dual.

Primal (Maximize) $\max Z = c^T x$	Dual (Minimize) $\min W = b^T y$
A: coefficient matrix	A^T: coefficient matrix
b: Right-hand-side vector	Cost vector
c: Price vector	Right-hand-side vector
ith constraint is an equation	The dual variable y_i has urs
ith constraint is $\leq$ type	The dual variable $y_i \geq 0$
ith constraint is $\geq$ type	The dual variable $y_i \leq 0$
x_j has urs	jth dual constraint is an equation
$x_j \geq 0$	jth dual constraint is $\geq$ type
$x_j \leq 0$	jth dual constraint is $\leq$ type

Example 1 Primal LP

$$\begin{aligned} & \max Z = x_1 + 4x_2 + 3x_3 \\ \text{subject to} \quad & 2x_1 + 3x_2 - 5x_3 \leq 2 \\ & 3x_1 - x_2 + 6x_3 \geq 1 \\ & x_1 + x_2 + x_3 = 4 \\ & x_1 \geq 0, \ x_2 \leq 0, \ x_3 \text{ urs.} \end{aligned}$$

Duel LP

$$\begin{aligned} & \min W = 2y_1 + y_2 + 4y_3 \\ \text{subject to} \quad & 2y_1 + 3y_2 + y_3 \geq 1 \\ & 3y_1 - y_2 + y_3 \leq 4 \\ & -5y_1 + 6y_2 + y_3 = 3 \\ & y_1 \geq 0, \ y_2 \leq 0, \ y_3 \text{ urs.} \end{aligned}$$

Example 2 Primal LP

$$\begin{aligned} & \min Z = 2x_1 + x_2 - x_3 \\ \text{subject to} \quad & x_1 + x_2 - x_3 = 1 \\ & x_1 - x_2 + x_3 \geq 2 \\ & x_2 + x_3 \leq 3 \\ & x_1 \geq 0, \ x_2 \leq 0, \ x_3 \text{ urs.} \end{aligned}$$

Duel LP

$$\begin{aligned} & \max W = y_1 + 2y_2 + 3y_3 \\ \text{subject to} \quad & y_1 + y_2 \leq 2 \\ & 3y_1 - y_2 + y_3 \geq 1 \\ & -y_1 + y_2 + y_3 = -1 \\ & y_1 \text{ urs}, \ y_2 \geq 0, \ y_3 \leq 0. \end{aligned}$$

Remark *The dual of the dual of an LP is the original problem.*

Theorem *Consider the standard primal and dual LP*

$$\max Z = c^T x,\ Ax \le b,\ x \ge 0 \quad \textit{and} \quad \min W = b^T y,\ A^T y \ge c,\ y \le 0$$

with $c \in \mathbb{R}^n, b \in \mathbb{R}^m, A \in \mathbb{R}^{m\times n}$. *If* $x_0 \in \mathbb{R}^n, y_0 \in \mathbb{R}^m$ *are vectors in the feasible regions so that* $Z^* = c^T x_0$ *and* $W^* = b^T y_0$ *are feasible solutions of the two problems, then* $Z^* \le W^*$.

As a result, if $Z^* = W^*$, *then it is the common optimal solutions for the primal and dual LP's.*

Furthermore, two column vectors $x_0 \in \mathbb{R}^n, y_0 \in \mathbb{R}^m$ *in the feasible regions will give rise to the optimal solution for the two problems if and only if*

$$(y_0^T A - c^T)x_0 + y_0^T(b - Ax_0) = 0, \quad \textit{i.e.,} \quad (y_0^T A - c^T)x_0 = y_0^T(b - Ax_0) = 0.$$

Proof. Let x_0 and y_0 be the vectors in the feasible regions giving rise to the Z^* and W^*. Then

$$Z^* = c^T x_0 \le (A^T y_0)^T x_0 = y_0^T A x_0 \le y_0^T b = b^T y_0 = W^*. \tag{1}$$

The second assertion is clear.

Finally, by (1), the equality $Z^* = W^*$ holds if and only if $Z^* = c^T x = y_0^T A x_0 = y_0^T b = W^*$, i.e., $(y_0^T A - c^T)x_0 = y_0^T(b - Ax_0) = 0$. The last assertion follows. □

The last condition is known as the complementary slackness principle for LP.

At the optimal solution: if $b - Ax_0$ has positive entries, i.e., the non-binding constraints, then the entries in y_0 equal zero; if $c - A^T c$ has positive entries, then the entries in x_0 equal zero.

Conversely, if we get two feasible solutions x_0, y_0 satisfying the condition

$$(y_0^T A - c^T)x_0 = y_0^T(b - Ax_0) = 0,$$

then $y_0^T A x_0$ is the optimal value for the two LP's.

Theorem *Consider the standard primal and dual problem. Exactly one of the following holds.*

(a) *If both problem are feasible, then both of them have optimal solutions having the same value.*

(b) *If one problem has unbounded solution, then the other problem has no feasible solution.*

(c) *Both problem are infeasible.*

Proof. Proof of (a) is tricky. Proof of (b) is easy. If (a) and (b) do not hold, then (c) holds. □

Note If P, D stand for the primal LP and dual LP.

(1) P has finite optimal if and only if D has finite optimal.

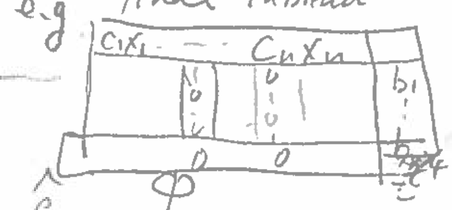
(2) if P is unbounded then D is infeasible;

(3) if D is unbounded then P is infeasible;

(4) if P is infeasible then D is unbounded or infeasible;

(5) if D is infeasible then P is unbounded or infeasible.

e.g. final tableau



$c - c_B B^{-1} A$