Dual Simplex Method.

Theorem *Consider the standard primal and dual LP*

$$\max Z = c^T x,\ Ax \le b,\ x \ge 0 \quad \textit{and} \quad \min W = b^T y,\ A^T y \ge c,\ y \le 0$$

with $c \in \mathbb{R}^n, b \in \mathbb{R}^m, A \in \mathbb{R}^{m\times n}$. If $x_0 \in \mathbb{R}^n, y_0 \in \mathbb{R}^m$ are vectors in the feasible regions so that $Z^* = c^T x_0$ and $W^* = b^T y_0$ are feasible solutions of the two problems, then $Z^* \le W^*$.

(1) *If $Z^* = W^*$, then it is the common optimal solutions for the primal and dual LP's.*

(2) *Two column vectors $x_0 \in \mathbb{R}^n, y_0 \in \mathbb{R}^m$ in the feasible regions will give rise to the optimal solution for the two problems if and only if*

$$(y_0^T A - c^T)x_0 + y_0^T(b - Ax_0) = 0, \quad \textit{i.e.,} \quad (y_0^T A - c^T)x_0 = y_0^T(b - Ax_0) = 0.$$

Proof. Let x_0 and y_0 be the vectors in the feasible regions giving rise to the Z^* and W^*. Then

$$Z^* = c^T x_0 \le (A^T y_0)^T x_0 = y_0^T A x_0 \le y_0^T b = b^T y_0 = W^*. \tag{1}$$

The second assertion is clear.

Finally, by (1), the equality $Z^* = W^*$ holds if and only if $Z^* = c^T x = y_0^T A x_0 = y_0^T b = W^*$, i.e., $(y_0^T A - c^T)x_0 = y_0^T(b - Ax_0) = 0$. The last assertion follows. $\square$

Condition (2) is known as the **complementary slackness principle** for LP. It can be rephrased as follows.

At the optimal solution:

- if $b - Ax_0 \in \mathbb{C}^m$ has positive entries, i.e., the non-binding constraints, then the entries in $y_0 \in \mathbb{R}^m$ equal zero;
- if $c - A^T y_0 \in \mathbb{R}^n$ has positive entries, then the entries in $x_0 \in \mathbb{R}^n$ equal zero.

Conversely, if we get two feasible solutions $x_0 \in \mathbb{R}^n, y_0 \in \mathbb{R}^m$ satisfying the condition

$$(y_0^T A - c^T)x_0 = y_0^T(b - Ax_0) = 0,$$

then $y_0^T A x_0$ is the optimal value for the two LP's.

Example 1 Primal LP

$$\begin{aligned} & \max Z = x_1 + 4x_2 + 3x_3 \\ \text{subject to} \quad & 2x_1 + 3x_2 - 5x_3 \le 2 \\ & 3x_1 - x_2 + 6x_3 \ge 1 \\ & x_1 + x_2 + x_3 = 4 \\ & x_1 \ge 0, \ \ x_2 \le 0, \ \ x_3 \text{ urs.} \end{aligned}$$

Duel LP

$$\begin{aligned} & \min W = 2y_1 + y_2 + 4y_3 \\ \text{subject to} \quad & 2y_1 + 3y_2 + y_3 \ge 1 \\ & 3y_1 - y_2 + y_3 \le 4 \\ & -5y_1 + 6y_2 + y_3 = 3 \\ & y_1 \ge 0, \ \ y_2 \le 0, \ \ y_3 \text{ urs.} \end{aligned}$$

In this example, $x_0 = (0, 0, 4)^T$ and $y_0 = (0, 0, 3)^T$ are feasible solutions such that

$$c^T x_0 = b^T y_0 = 12$$

is the optimal for both the primal and dual problems.

Clearly, the complementary slackness conditions holds:

For $A = \begin{pmatrix} 2 & 3 & -5 \\ 3 & -1 & 6 \\ 1 & 1 & 1 \end{pmatrix}$, $b - Ax_0 = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} - 4 \begin{pmatrix} -5 \\ 6 \\ 1 \end{pmatrix} = \begin{pmatrix} 22 \\ -23 \\ 0 \end{pmatrix}$, and

$$y_0^T A - c = 3(1, 1, 1) - (1, 4, 3) = (2, -1, 0).$$

Example 2 Primal LP

$$\begin{aligned} & \min Z = 2x_1 + x_2 - x_3 \\ \text{subject to} \quad & x_1 + x_2 - x_3 = 1 \\ & x_1 - x_2 + x_3 \ge 2 \\ & x_2 + x_3 \le 3 \\ & x_1 \ge 0, \ \ x_2 \le 0, \ \ x_3 \text{ urs.} \end{aligned}$$

Duel LP

$$\begin{aligned} & \max W = y_1 + 2y_2 + 3y_3 \\ \text{subject to} \quad & y_1 + y_2 \quad \le 2 \\ & 3y_1 - y_2 + y_3 \ge 1 \\ & -y_1 + y_2 + y_3 = -1 \\ & y_1 \text{ urs}, \ \ y_2 \ge 0, \ \ y_3 \le 0. \end{aligned}$$

In this example, $x_0 = (2, 0, 1)^T$ and $y_0 = (1, 0, 0)^T$ are feasible solutions such that

$$c^T x_0 = 3 > 1 = b^T y_0.$$

The two LP's should have a finite optimal solution assuming the same value.

The dual simplex method

Theorem *Consider the standard primal and dual problem. Exactly one of the following holds.*

(a) *If both problem are feasible, then both of them have optimal solutions having the same value.*

(b) *If one problem has unbounded solution, then the other problem has no feasible solution.*

(c) *Both problem are infeasible.*

Proof. Proof of (a) is tricky. Proof of (b) is easy. If (a) and (b) do not hold, then (c) holds. □

Note If P, D stand for the primal LP and dual LP.

(1) P has finite optimal if and only if D has finite optimal.

(2) if P is unbounded then D is infeasible;

(3) if D is unbounded then P is infeasible;

(4) if P is infeasible then D is unbounded or infeasible;

(5) if D is infeasible then P is unbounded or infeasible.

Solving the primal LP to get the solution for the dual LP

Consider the primal problem in standard form

$$\max Z = c^T x \qquad \text{subject to} \quad Ax = b, \quad x \geq 0.$$

The dual LP has the form

$$\min W = b^T y, \qquad \text{subject to} \quad A^T y \leq c, \quad \text{all entries of } y \text{ has urs}.$$

Note that if x_0 is an basic feasible optimal solution, then $\bar{C} = c^T - c_B^T B^{-1} A \leq 0$. If $y^T = c_B^T B^{-1}$, then

$$c^T \geq y^T A \qquad \text{and} \qquad W = y^T b = c_B^T B^{-1} b = Z.$$

So, $Z = W$ is the optimal solution of the LP's; $y = c_B^T B^{-1}$ is an optimal solution for the duel LP.

Example Primal LP

$$
\begin{array}{lrl}
 & \min Z = -32x_1 + x_2 + x_3 & \\
\text{subject to} & x_1 - 2x_2 + x_3 + x_4 \qquad & = 11 \\
 & -4x_1 + x_2 + 2x_3 \qquad - x_5 & = 3 \\
 & -2x_1 \qquad + x_3 \qquad & = 1 \\
 & x_1, x_2, x_3, x_4, x_5 & \geq 0
\end{array}
$$

Duel LP

$$
\begin{array}{lrl}
 & \max W = 11y_1 + 3y_2 + y_3 & \\
\text{subject to} & y_1 - 4y_2 - 2y_3 & \leq -3 \\
 & -2y_1 + y_2 & \leq 1 \\
 & y_1 + 2y_2 + y_3 & \leq 1 \\
 & y_1 & \leq 0 \\
 & -y_2 & \leq 0 \\
 & y_1, y_2, y_3 \text{ urs.} &
\end{array}
$$

We can solve the primal LP to get $(x_1, x_2, x_3) = (4, 1, 9)$ with $Z = -2$.
Then $y = c_B B^{-1} = (-3, 1, 1)B^{-1} = (-1, 1, 2)/3$ is the dual optimal solution.

See the matlab commands file

1. We use two phase method to solve the primal LP
2. We can use matlab command to solve the primal LP

Once the primal LP is done

Compute $y = c_B B^{-1}$ and check that it is the dual solution.

Recall: 96 $\left[\begin{array}{c|c} A & b \\ \hline c & \end{array}\right]$ is the original tableau.

and $x_{i_1} \cdots x_{i_m}$ are the basic variables for optimal, then $B = [A_{i_1} \cdots A_{i_m}]$ will yield the final tableau

$$\left[B^{-1}A \mid B^{-1}b \right]$$

Duel Simplex Method: Solving the dual LP to get the solution for the primal LP

If one solves the primal LP max $Z = c^T x$ subject to $Ax \le b$, $x \ge 0$, and get an basic feasible optimal solution, the $y = c_B B^{-1}$ is a optimal solution for the dual LP min $Z = b^T y$ subject to $A^T y \ge c$, entries of y have unrestricted signs.

If we run into a situation that

$$\tilde{C} = c^T - c_B^T B^{-1} A \ge 0,$$

then we have the dual feasibility vector y with $y^T = c_B^T B^T$.

Case 1. If it corresponds to a primal feasible vector x, we are done.

Case 2. If not, apply the simplex algorithm to the dual problem (in the same tableau) as follows.

Step 1. Choose $\tilde{b} = B^{-1} b$ with the most negative value (shadow price), say, $\tilde{b}_r$.

Step 2. Check whether there is $\tilde{a}_{rj}$ in $\tilde{A} = B^{-1} A$ with negative coefficients. If no, the primal problem is infeasible. If yes, select $\tilde{a}_{rj}$ such that $c_j / \tilde{a}_{rj}$ is maximum among those j with $\tilde{a}_{rj} < 0$.

Example min $Z = x_1 + 4x_2 \qquad + 3x_4$

Subject to:

$$x_1 + 2x_2 - x_3 + x_4 \ge 3$$
$$-2x_1 - x_2 + 4x_3 + x_4 \ge 2$$
$$x_1, x_2, x_3, x_4 \ge 0.$$

Use excess variables x_5, x_6 to get the standard form

$$\min Z = x_1 + 4x_2 \qquad + 3x_4$$

Subject to:

$$x_1 + 2x_2 - x_3 + x_4 - x_5 \qquad \ge 3$$
$$-2x_1 - x_2 + 4x_3 + x_4 \qquad - x_6 \ge 2$$
$$x_1, x_2, x_3, x_4, x_5, x_6 \ge 0.$$

C_B	B	$(1)x_1$	$(+4)x_2$	$(+0)x_3$	$(+3)x_4$	$(+0)x_5$	$(+0)x_6$	constraints
0	x_5	-1*	-2	1	-1	1	0	-3
0	x_6	2	1	-4	-1	0	1	-2
	$\tilde{C}$	1	4	0	3	0	0	

Here, we choose x_1 because of the ratio of $(-1, -2, -1)$ to $(1, 4, 3)$ equals $(-1, -2, -3)$.

C_B	B	$(1)x_1$	$(+4)x_2$	$(+0)x_3$	$(+3)x_4$	$(+0)x_5$	$(+0)x_6$	constraints
1	x_1	1	2	-1	1	-1	0	3
0	x_6	0	-3	-2*	-3	2	1	-8
	$\tilde{C}$	0	2	1	2	1	0	

Here we choose x_3 because the ratio of $(-3, -2, -3)$ to $(2, 1, 2)$ is $(-2/3, -1/2, -2/3)$.

C_B	B	$(1)x_1$	$(+4)x_2$	$(+0)x_3$	$(+3)x_4$	$(+0)x_5$	$(+0)x_6$	constraints
1	x_1	1	$7/2$	0	$5/2$	-2	$-1/2$	7
0	x_3	0	$3/2$	1	$3/2$	-1	$-1/2$	4
	$\tilde{C}$	0	$1/2$	0	$1/2$	2	$1/2$	$Z = 7$

For minimization problem, This is the optimal condition for the primial problem. i.e. feasible condition for the dual.

Back to the examples in sensitivity analysis.

Example

C_B	B	$(+2)x_1$	$(+3)x_2$	$(+1)x_3$	$(0)x_4$	$(+1)x_5$	constraints
2	x_1	1	0	-1	4	-1	13
3	x_2	0	1	2	-1^*	1	-1
	C	0	0	-3	-5	-1	

C_B	B	$(+2)x_1$	$(+3)x_2$	$(+1)x_3$	$(0)x_4$	$(+1)x_5$	constraints
2	x_1	1	4	7	0	3	9
0	x_4	0	-1	-2	1	-1	1
	C	0	-5	-13	0	-6	$Z = 18$

Example

C_B	B	$(+2)x_1$	$(+3)x_2$	$(+1)x_3$	$(+0)x_4$	$(+1)x_5$	$(+0)x_6$	constraints
2	x_1	1	0	-1	4	-1	0	1
3	x_2	0	1	2	-1	1	0	2
0	x_5	0	0	-2	-2	-1^*	1	-1
	C	0	0	-3	-5	-1	0	

C_B	B	$(+2)x_1$	$(+3)x_2$	$(+1)x_3$	$(+0)x_4$	$(+1)x_5$	$(+0)x_6$	constraints
2	x_1	1	0	1	6	0	-1	2
3	x_2	0	1	0	-3	0	1	1
0	x_6	0	0	2	2	1	-1	1
	C	0	0	-1	-3	0	-1	$Z = 7$

For maximization problem, these are the optimal conditions for the primal. i, the feasible conditions for the dual problem