Transportation, Assignment, and Transshipment problem

7.1 Transportation problem

Example Powerco has 3 plants $P1, P2, P3$ that can supply powers 35, 50, 45 million of kilowatts, respectively, to 4 cities $C1, C2, C3, C4$ with demands of 45, 20, 30, 30 million of kilowatts. Shipping costs, demands, and supplies constraints are summarized in the following table.

	C1	C2	C3	C4	Supply
P1	\$8	\$6	\$10	\$9	≤ 35
P2	\$9	\$12	\$13	\$7	≤ 50
P3	\$14	\$9	\$16	\$5	≤ 40
Demand	≥ 45	≥ 20	≥ 30	≥ 30	

125

125

Suppose x_{ij} is the number of kilowatts from Pi to Cj, and let c_{ij} be the cost (in \$) from Pi to Cj.

The LP problem: $\min Z = \sum_{i,j} c_{ij}x_{ij}$

Subject to

$$\begin{aligned}
x_{11} + x_{12} + x_{13} + x_{14} &\leq 35 && \text{(P1 supply constraint)}\\
x_{21} + x_{22} + x_{23} + x_{24} &\leq 50 && \text{(P2 supply constraint)}\\
x_{31} + x_{32} + x_{33} + x_{34} &\leq 40 && \text{(P3 supply constraint)}\\
x_{11} + x_{21} + x_{31} &\geq 45 && \text{(C1 demand constraint)}\\
x_{12} + x_{22} + x_{32} &\geq 20 && \text{(C2 demand constraint)}\\
x_{13} + x_{23} + x_{33} &\geq 30 && \text{(C3 demand constraint)}\\
x_{14} + x_{24} + x_{34} &\geq 30 && \text{(C4 demand constraint)}\\
x_{ij} &\geq 0.
\end{aligned}$$

General Formulation Assume there are m suppliers shipping a certain product to n stores such that the cost of shipment from Si (ith supply point) to Dj (jth demand point) is c_{ij}. We need to solve the LP:

$$\text{min}/\max Z = \sum_{i,j} c_{ij}x_{ij}$$

subject to

$$\begin{aligned}
\textstyle\sum_j x_{ij} &\leq s_i \quad \text{for } i = 1, \dots, m, && \text{(supply constraints)}\\
\textstyle\sum_i x_{ij} &\geq d_j \quad \text{for } j = 1, \dots, n, && \text{(demand constraints)}\\
x_{ij} &\geq 0.
\end{aligned}$$

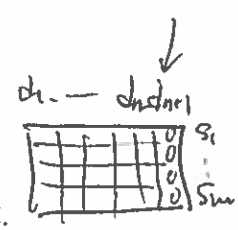
Remarks

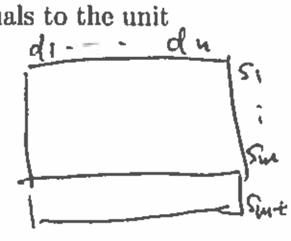
1. The transportation company may want to solve the max $Z = \sum_{i,j} c_{ij}x_{ij}$.
2. The problem is $\boxed{\text{balanced}}$ if $\sum_i s_i = \sum_j d_j$.
3. If $\sum_i s_i > \sum_j d_j$, we may set up a dummy demand $\boxed{d_{n+1}}$ with costs $c_{i,n+1} = 0$ for all i.

   $// \sum_{i=1}^{m} s_i - \sum_{j=1}^{n} d_j$

4. If $\sum_i s_i < \sum_j d_j$, we may set up a dummy supplier s_{m+1} with costs $c_{m+1,j}$ equals to the unit penalty amount imposed by Dj.

   $// \sum_{j=1}^{n} d_j - \sum_{i=1}^{m} s_i$





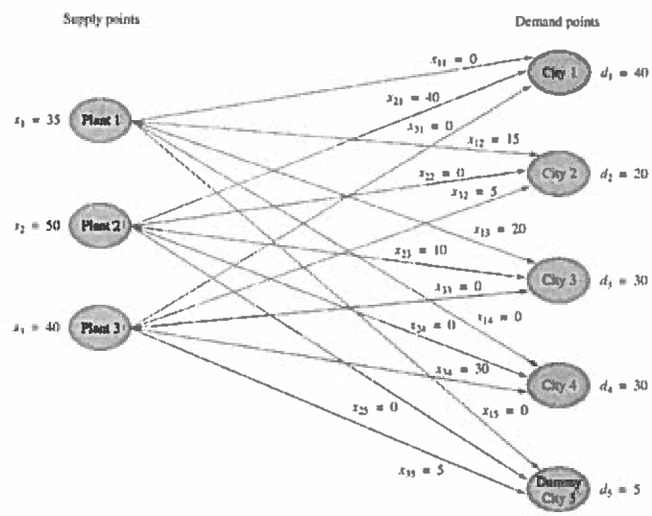


FIGURE 2
Graphical Representation of Unbalanced Powerco Problem and Its Optimal Solution (with Dummy Demand Point)

TABLE 2
A Transportation Tableau

				Supply
c_{11}	c_{12}	...	c_{1n}	s_1
c_{21}	c_{22}	...	c_{2n}	s_2
⋮	⋮		⋮	⋮
c_{m1}	c_{m2}	...	c_{mn}	s_m
d_1	d_2	...	d_n	

Demand

TABLE 3
Transportation Tableau for Powerco

	City 1	City 2	City 3	City 4	Supply
Plant 1	8	6 10	10 25	9	35
Plant 2	9 45	12	13 5	7	50
Plant 3	14	9 10	16	5 30	40
Demand	45	20	30	30	

Note:

x_{11} - - - - x_{34}

1 1 1 1 1 1 1 1 1 1 1 1 | s_1 s_2 s_3

I_3 I_3 I_3 I_3 | d_1 d_2 d_3 d_4

Only 6 → 7

m+n-1 independent equations

So we can delete one of the row to get the initial tableau. and every basic feasible solution requires (m+n-1) variables

Inventory problem as transportation problem

Example Sailco manufactures sailboats.

Demands for the next 4 quarters are: 40, 60, 75, 25.

At the beginning, there are 10 sailboats in inventory.

Each quarter, have to make 40 sailboats at the cost of $400 each.

Additional sailboat can be made at a cost of $450 each.

Left over inventory cost $20 per sailboat for each quarter.

We can formulate the following transportation problem.

Supply points.

S1 inventory (s1 = 10)
S2 quarter 1 regular production (s2= 40)
S3 quarter 1 overtime production (s3 = 150)
S4 quarter 2 regular production (s4= 40)
S5 quarter 2 overtime production (s5 = 150)
S6 quarter 3 regular production (s6= 40)
S7 quarter 3 overtime production (s7 = 150)
S8 quarter 4 regular production (s8= 40)
S9 quarter 4 overtime production (s9 = 150)

Here, regular production must be 40 per quarter. Overtime production has no limit, the total demand is 200, subtracting the initial inventory 10, and 40 regular production in the first quarter, so the maximum should be 150.

Demand points

D1 quarter 1 demand (d1 = 40)
D2 quarter 2 demand (d2 = 60)
D3 quarter 3 demand (d3 = 75)
D4 quarter 4 demand (d4 = 25)
D5 dummy demand (d5 = 700-200 = 570)

See the next page for the costs.

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TABLE 6
Transportation Tableau for Sailco

	1	2	3	4	Dummy	Supply
Initial	0 / 10	20	40	60	0	10
Qtr 1 RT	400 / 30	420 / 10	440	460	0	40
Qtr 1 OT	450	470	490	510	0 / 150	150
Qtr 2 RT	M	400 / 40	420	440	0	40
Qtr 2 OT	M	450 / 10	470	490	0 / 140	150
Qtr 3 RT	M	M	400 / 40	420	0	40
Qtr 3 OT	M	M	450 / 35	470	0 / 115	150
Qtr 4 RT	M	M	M	400 / 25	0 / 15	40
Qtr 4 OT	M	M	M	450	0 / 150	150
Demand	40	60	75	25	570	

7.2 Finding a basic feasible solution

For a balanced transportation problem, there are mn variables x_{ij}, and $m+n-1$ linearly independent equalities.

1. To form a bfs, one needs to choose $m+n-1$ variables x_{ij}.
2. Arbitrary choices of $m+n-1$ variables may not correspond to a basic feasible solution.
3. The selection of those $\{x_{ij}\}$ do not contain a loop. That is, it contains a sequences

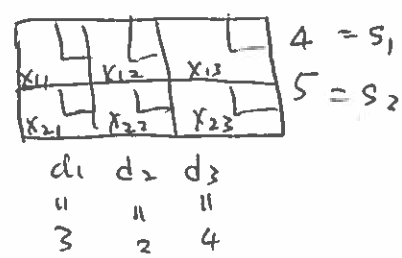
$$x_{i_1,j_1}, x_{i_1,j_2}, x_{i_2,j_2}, x_{i_2,j_3}, \ldots, x_{i_k,j_k}, x_{i_k,j_1}$$

so that $i_1, \ldots, i_k$ are distinct, and $j_1, \ldots, j_k$ are distinct.

Example $(s_1, s_2) = (4,5), (d_1, d_2, d_3) = (3,2,4)$, $(x_{11}, x_{12}, x_{21}, x_{22})$ cannot form a bsf.

Reason. We have to solve

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \\ x_{21} \\ x_{22} \\ x_{23} \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 3 \\ 2 \\ 4 \end{bmatrix}.$$



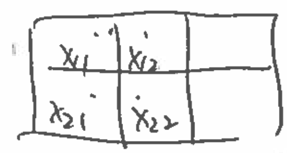
Removing a redundant equality, we have to solve

$$\begin{array}{cccccc} x_{11} & x_{12} & x_{13} & x_{21} & x_{22} & x_{23} \\ \phi & \phi & & \phi & \phi & \end{array}$$

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \\ x_{21} \\ x_{22} \\ x_{23} \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 2 \\ 4 \end{bmatrix}.$$

If $(x_{11}, x_{12}, x_{21}, x_{22})$ yields a bsf, then

$$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \\ x_{21} \\ x_{22} \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 2 \\ 4 \end{bmatrix},$$



which is impossible.

$x_{i_1j_1}$ $x_{i_1j_2}$ $x_{i_2j_2}$ $x_{i_2j_3}$