Three ways of finding basic feasible solutions

1. Northwest corner method.

   From the $(1,1)$ entry, try to fulfill the row or column sum constraint in each step.

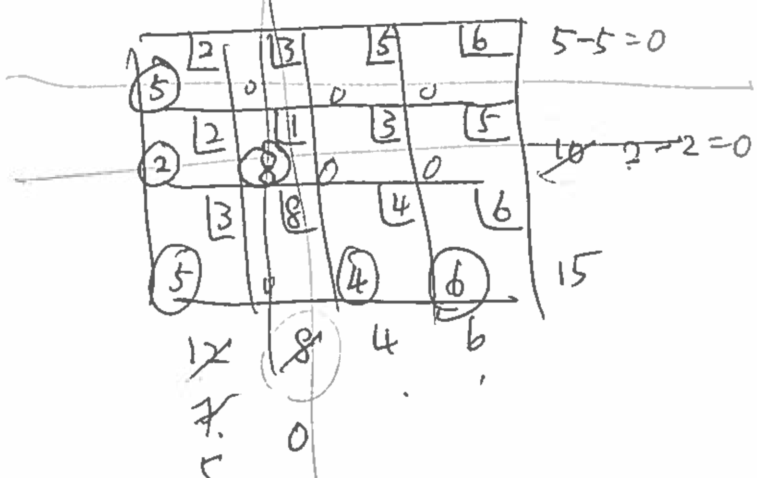
   Example: $(s_1, s_2, s_3) = (5, 1, 3), (d_1, d_2, d_3, d_4) = (2, 4, 2, 1)$.

2	3	0	0	5
0	1	0		1
0		2	1	3
2	4	2	1	

2. Minimum cost method.

   Use the cheapest cost in each step to satisfy the row or column in each step.

   Example: $(s_1, s_2, s_3) = (5, 10, 15), (d_1, d_2, d_3, d_4) = (12, 8, 4, 6)$, $C = \begin{pmatrix} 2 & 3 & 5 & 6 \\ 2 & 1 & 3 & 5 \\ 3 & 8 & 4 & 6 \end{pmatrix}$.



3. Vogel's method. Choose cheap costs and avoid future heavy penalty.

   Compute row/column penalties (difference of the two minimum costs in each row/column).

   Select basic variable at the row or column with maximum penalty.

TABLE 28

				Supply	Row Penalty
	$\boxed{6}$	$\boxed{7}$	$\boxed{8}$	10	7 − 6 = 1
	$\boxed{15}$	$\boxed{80}$	$\boxed{78}$	15	78 − 15 = 63
Demand	15	5	5		
Column Penalty	15 − 6 = 9	80 − 7 = 73	78 − 8 = 70		

TABLE 29

				Supply	Row Penalty
	$\boxed{6}$	5 $\boxed{7}$	$\boxed{8}$	5	8 − 6 = 2
	$\boxed{15}$	$\boxed{80}$	$\boxed{78}$	15	78 − 15 = 63
Demand	15	×	5		
Column Penalty	9	—	70		

TABLE 30

				Supply	Row Penalty
	$\boxed{6}$	5 $\boxed{7}$	5 $\boxed{8}$	0	—
	$\boxed{15}$	$\boxed{80}$	$\boxed{78}$	15	—
Demand	15	×	×		
Column Penalty	9	—	—		

TABLE 31

				Supply	Row Penalty
	0 $\boxed{6}$	5 $\boxed{7}$	5 $\boxed{8}$	×	—
	$\boxed{15}$	$\boxed{80}$	$\boxed{78}$	15	—
Demand	15	×	×		
Column Penalty	—	—	—		

$c^T x$

max $z =$ $Ax \le b$

min $W = b^T y$

$A^T y \le c$

7.3 The transportation simplex method

1. Set up the balanced transportation problem with m supply points and n demand points to minimize $Z = \sum_{i,j} c_{ij} x_{ij}$.
2. Find an initial basic feasible solution.
3. Find $(u_1, \ldots, u_m, v_1, \ldots, v_n)$ with $u_1 = 0$ and $u_i + v_j = c_{ij}$ for those c_{ij} corresponding to the basic variables x_{ij}.

   min $Z = \sum c_{ij} x_{ij}$

   $A x = \begin{bmatrix} s_1 \\ s_2 \\ s_m \\ d_1 \\ \vdots \\ d_n \end{bmatrix}$

   Note that $(u_1, \ldots, u_m, v_1, \ldots, v_n)$ is a "proposed" solution of the dual LP problem:

   $$\max W = \sum_{i=1}^{m} s_i u_i + \sum_{j=1}^{n} d_j v_j \qquad \text{Subject to } A^t \begin{bmatrix} u \\ v \end{bmatrix} \le \begin{bmatrix} c_{11} \\ \vdots \\ c_{mn} \end{bmatrix},$$

   $u = [u_1, \ldots, u_m]^t$, $v = [v_1, \ldots, v_n]^t$ have entries with unrestricted signs.
4. If $u_i + v_j \le c_{ij}$ for all (i, j) pairs, then $(u_1, \ldots, u_m, v_1, \ldots, v_n)$ is dual feasible. So, we get an optimal solution.
5. Otherwise, choose the (i, j) pair such that $\boxed{u_i + v_j - c_{ij} > 0}$ is maximum to be the entering variable.
6. Find a (the) loop using x_{rs} in the basic feasible solutions together with x_{ij}, and use x_{ij} as entry 0 in the loop.
7. Find the maximum $\delta > 0$ to add to the the even entries x_{rs} in the loop, and subtract δ from the odd entries in the loop.

   [An odd entries x_{rs} in the loop that is reduced to 0 after the procedure is the basic variable changing into a non-basic variable (as x_{ij} becomes a basic variable).]
8. Go back to Step 3 until an optimal solution (both primal and dual feasible) is found.

Remark For the maximization problem max $Z = \sum_{i,j} c_{ij} x_{ij}$, the dual problem is:

$$\min W = \sum_{i=1}^{m} s_i u_i + \sum_{j=1}^{n} d_j v_j \qquad \text{Subject to } A^t \begin{bmatrix} u \\ v \end{bmatrix} \ge \begin{bmatrix} c_{11} \\ \vdots \\ c_{mn} \end{bmatrix},$$

$u = [u_1, \ldots, u_m]^t$, $v = [v_1, \ldots, v_n]^t$ have entries with unrestricted signs.

So, we modify (4), (5) to:

4' The current solution is optimal if the proposed solution $(u_1, \ldots, u_m, v_1, \ldots, v_n)$ of the dual problem satisfies $u_i + v_j \ge c_{ij}$ for all (i, j).

5' Otherwise, find the (i, j) pair such that $c_{ij} - (u_i + v_j) > 0$ is maximum to be the entering variable.

Example Solve the Powerco problem.

	C1	C2	C3	C4	Supply
P1	\$8	\$6	\$10	\$9	≤ 35
P2	\$9	\$12	\$13	\$7	≤ 50
P3	\$14	\$9	\$16	\$5	≤ 40
Demand	≥ 45	≥ 20	≥ 30	≥ 30	

$v_1=8$ $v_2=11$ $v_3=12$ $v_4=1$

$0 = u_1$

$1 = u_2$

$4 = u_3$

	City1	City2	City3	City4	Supply
Plant1	0 8 35−Δ	5 6	2 10	−8 9 +Δ	35
Plant2	0 9 10+Δ	0 12 20	0 13 20−Δ	−5 7	50
Plant3	−2 14	6 9	0 16 10+Δ	0 5 30−Δ	40
Demand	45	20	30	30	

$u_1 =$ $v_1 =$

$u_2 =$ $v_2 =$

$u_3 =$ $v_3 =$

$v_4 =$

$\Delta = 20$

	City1	City2	City3	City4	Supply
Plant1	8 15	6	10	9 20	35
Plant2	9 30	12 20	13	7	50
Plant3	14	9	16 30	5 10	40
Demand	45	20	30	30	

$u_1 =$ $v_1 =$

$u_2 =$ $v_2 =$

$u_3 =$ $v_3 =$

$v_4 =$

$\Delta =$

	City1	City2	City3	City4	Supply
Plant1	8	6	10	9	35
Plant2	9	12	13	7	50
Plant3	14	9	16	5	40
Demand	45	20	30	30	

$u_1 =$ $v_1 =$

$u_2 =$ $v_2 =$

$u_3 =$ $v_3 =$

$v_4 =$

$\Delta =$

	City1	City2	City3	City4	Supply
Plant1	8	6	10	9	35
Plant2	9	12	13	7	50
Plant3	14	9	16	5	40
Demand	45	20	30	30	

$u_1 =$ $v_1 =$

$u_2 =$ $v_2 =$

$u_3 =$ $v_3 =$

$v_4 =$

$\Delta =$