Math 323 Operations Research **Notes on Chapter 8** C.K. Li

Network models

Basic definitions

A graph or a network consists of a vertex set $\boxed{V = \{v_1, \ldots, v_n\}}$ and a set of arcs A containing a selection of order pairs (v_i, v_j) of vertices. The vertex v_i is the initial node, and the vertex if the terminal node of the arc.

A chain is a sequence of arcs such that each arc (starting from the second one) has one vertex in common with the previous arc.

A path is a chain such that (each arc starting from the second one) has the its initial node equal to the terminal node of the previous arc.

$V = \{1, 2, 3, 4\}$

$A = \{(1,2), (2,3), (3,4), (4,1)$
$(4,3)\}$

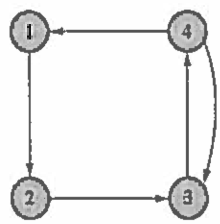


FIGURE 1
Example of a Network

$\boxed{N = (V, A)}$

network vertex set Arc Set

Paths: $(1,2), (2,3), (3,4)$

Chain: $(4,1), (4,3), (2,3)$

$(1,2), (3,4)$

X path

X chain

8.2 Shortest path problems

Shortest path problem. Given a network so that every arc is associated with a distance. Find a path from vertex i to vertex t with shortest distance, i.e., the sum of distance of the arcs of the path is minimum.

Dijkstra's algorithm

1. List the vertices $[1, \ldots, n]$ with 1 as the initial vertex, n is the final vertex, and set assign a permanent label 0 to the initial vertex 1.

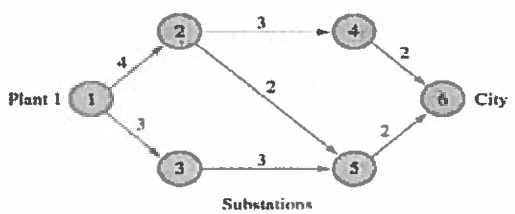
2. Assign temporary label to each other vertex j as the cost of the arc $(1, j)$, which is ∞ if $(1, j)$ is not an arc. Select a vertex with minimum label to be permanent.

3. Suppose k is the permanent vertex labeled most recently. Update the label of each non-permanent vertex j using the minimum of the original label of j and the sum of label k and the cost from k to j, which is ∞ if (j, k) is not an arc.

4. Select the undated temporary label with minimum cost as a new permanent labeled vertex. If it is the terminal vertex, we are done. Back track the permanent label to recover the shortest path. Else, go back to Step 3.

Example

FIGURE 2
Network for Powerco



$1 \downarrow 2 \downarrow 5 \downarrow 6$

$1 \downarrow 3 \downarrow 5 \downarrow 6$

$[0^* \quad 4 \quad 3 \quad \infty \quad \infty \quad \infty]$

$[0^* \quad 4 \quad 3^* \quad \infty \quad \infty \quad \infty]$

$[0^* \quad 4^* \quad 3^* \quad \infty \quad 6 \quad \infty]$

$[0^* \quad 4^* \quad 3^* \quad 7 \quad 6^* \quad \infty]$

$[0^* \quad 4^* \quad 3^* \quad 7^* \quad 6^* \quad 8]$

$[0^* \quad 4^* \quad 3^* \quad 7^* \quad 6^* \quad 8^*]$

Remark We can formulate the shortest path problem as a transshipment problem.

- Set each vertex except the final one as a supply point with supply value 1.
- Set each vertex except the initial one as a demand point with demand value 1.
- Assign cost c_{ij} from supply point i to demand point j.
- Find the minimum cost for the transportation problem.

TABLE 3
Transshipment Representation of Shortest-Path Problem and Optimal Solution (1)

Node	2	3	4	5	6	Supply
1	4; 1	3	M	M	M	1
2	0	M	3	2; 1	M	1
3	M	0; 1	M	3	M	1
4	M	M	0; 1	M	2	1
5	M	M	M	0	2; 1	1
Demand	1	1	1	1	1	

To illustrate the preceding ideas, we formulate the balanced transportation problem associated with finding the shortest path from node 1 to node 6 in Figure 2. We want to send one unit from node 1 to node 6. Node 1 is a supply point, node 6 is a demand point, and nodes 2, 3, 4, and 5 will be transshipment points. Using $s = 1$, we obtain the balanced transportation problem shown in Table 3. This transportation problem has two optimal solutions:

1 $z = 4 + 2 + 2 = 8$, $x_{12} = x_{25} = x_{56} = x_{33} = x_{44} = 1$ (all other variables equal 0). This solution corresponds to the path 1–2–5–6.

2 $z = 3 + 3 + 2 = 8$, $x_{13} = x_{35} = x_{56} = x_{22} = x_{44} = 1$ (all other variables equal 0). This solution corresponds to the path 1–3–5–6.

Related problem - equipment replacement. Suppose an equipment is purchased. There are repair cost, trade in cost, etc.

Find the minimum net purchase cost = purchase cost + maintenance cost − trade in price.

Example Buying a new car costs \$12000 For instance, annual maintenance cost (M) for a car of n year old, and the trade in (T) price for a car of n year old.

n	0	1	2	3	4	5
M	2000	4000	5000	9000	12000	
T		7000	6000	2000	1000	0

Set up a network with vertices $V = \{1, \ldots, 6\}$. The arc from (i, j) with $i < j$ has a cost

c_{ij} = buying price of a car + maintenance cost from year i to $j - 1$ - trade in cost in year.

We can build up the network and find the "shortest path".

$c_{12} = 2 + 12 - 7 = 7$

$c_{13} = 2 + 4 + 12 - 6 = 12$

$c_{14} = 2 + 4 + 5 + 12 - 2 = 21$

$c_{26} = 2 + 4 + 5 + 9 + 12 - 1 = 31$

$c_{34} = 2 + 12 - 7 = 7$

$c_{35} = 2 + 4 + 12 - 6 = 12$

$c_{36} = 2 + 4 + 5 + 12 - 2 = 21$

$c_{16} = 2 + 4 + 5 + 9 + 12 + 12 - 0 = 44$

$c_{23} = 2 + 12 - 7 = 7$

$c_{24} = 2 + 4 + 12 - 6 = 12$

$c_{45} = 2 + 12 - 7 = 7$

$c_{46} = 2 + 4 + 12 - 6 = 12$

$c_{56} = 2 + 12 - 7 = 7$

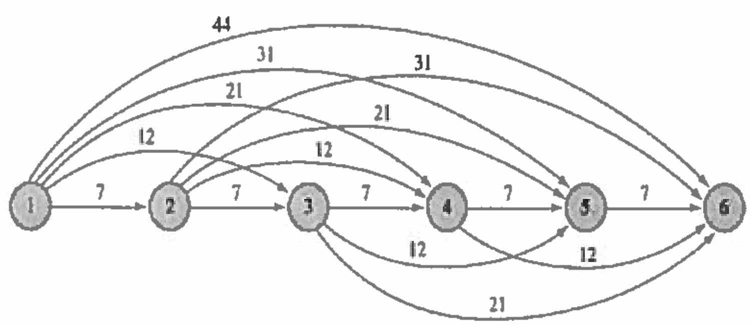


FIGURE 3
Network for Minimizing Car Costs

8.3 Maximum flow - minimum cut problems

Problem In a network so that each arc (i, j) has a flow capacity c_{ij} constraint. Find the maximum flow value from a source vertex so to a sink vertex si.

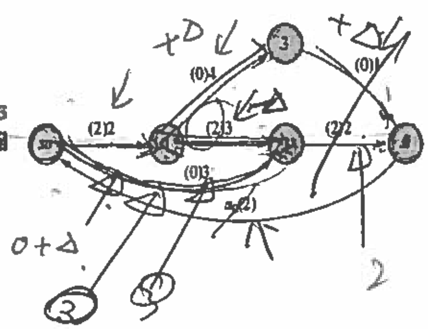
EXAMPLE 3 Maximum Flow

Sunco Oil wants to ship the maximum possible amount of oil (per hour) via pipeline from node *so* to node *si* in Figure 6. On its way from node *so* to node *si*, oil must pass through some or all of stations 1, 2, and 3. The various arcs represent pipelines of different diameters. The maximum number of barrels of oil (millions of barrels per hour) that can be pumped through each arc is shown in Table 8. Each number is called an **arc capacity.** Formulate an LP that can be used to determine the maximum number of barrels of oil per hour that can be sent from *so* to *si*.

TABLE 8
Arc Capacities for Sunco Oil

Arc	Capacity
$(so, 1)$	2
$(so, 2)$	3
$(1, 2)$	3
$(1, 3)$	4
$(3, si)$	1
$(2, si)$	2

FIGURE 6
Network for Sunco Oil



$$
\begin{aligned}
\max z &= x_0 \\
\text{s.t.} \quad x_{so,1} &\le 2 && \text{(Arc capacity constraints)} \\
x_{so,2} &\le 3 \\
x_{12} &\le 3 \\
x_{2,si} &\le 2 \\
x_{13} &\le 4 \\
x_{3,si} &\le 1 \\
x_0 &= x_{so,1} + x_{so,2} && \text{(Node } so \text{ flow constraint)} \\
x_{so,1} &= x_{12} + x_{13} && \text{(Node 1 flow constraint)} \\
x_{so,2} + x_{12} &= x_{2,si} && \text{(Node 2 flow constraint)} \\
x_{13} &= x_{3,si} && \text{(Node 3 flow constraint)} \\
x_{3,si} + x_{2,si} &= x_0 && \text{(Node } si \text{ flow constraint)} \\
x_{ij} &\ge 0
\end{aligned}
$$

One optimal solution to this LP is $z = 3$, $x_{so,1} = 2$, $x_{13} = 1$, $x_{12} = 1$, $x_{so,2} = 1$, $x_{3,si} = 1$, $x_{2,si} = 2$, $x_0 = 3$. Thus, the maximum possible flow of oil from node *so* to *si* is 3 million barrels per hour, with 1 million barrels each sent via the following paths: *so*–1–2–*si*, *so*–1–3–*si*, and *so*–2–*si*.

Remark We can always set up the maximum flow problem as a transportation problem:

$$\max Z = x_0$$

subject to: $\sum_i f_{ij} - \sum_k f_{jk} = 0, \quad j = 1, \ldots, n.$

$$\sum f_{so,i} = x_0, \qquad \sum f_{j,si} = x_0, \qquad 0 \le f_{ij} \le c_{ij}.$$

Ford-Fulkerson Algorithm

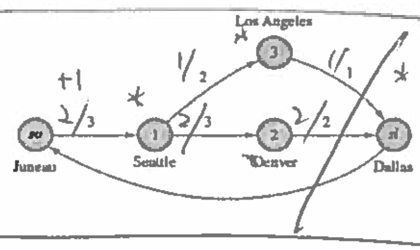
Consider a capacitated network with source vertex *so* and sink vertex *si*. Partition $V = \boxed{S \cup \overline{S}}$ with $so \in S, si \in \overline{S}$. Define the cut associated with $(S, \overline{S})$ as $K(S, \overline{S}) = \sum_{(i,j) \in (S \times \overline{S})} c_{ij}$. Then maximum flow in the network equals the minimum cut.

1. Set initial flow to be 0.
2. Find a chain from *so* to *si* consisting of non-saturated forward arcs $c_{ij} - f_{ij} > 0$, and backward arcs with non-zero flows f_{rs}; increase the flow by the minimum of the values $c_{ij} - f_{ij}$ and f_{rs}.

   This can be done by labeling the vertices starting from *so*; after adding a new round of newly labeled vertices, move on to the next round by labeling those vertices connected with those labeled in the last round by forward non-saturated forward arcs or and backward arcs with positive flow until we reach *si*, or find it impossible.
3. If no such chain exists, then we have an optimal flow. (Letting S be the vertices reachable from *so* with a positive chain. Then $K(S, \overline{S})$ is a minimum cut.)

Example

FIGURE 7
Network for Fly-by-Night Airlines



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Solution The appropriate network is given in Figure 7. Here the capacity of arc (i, j) is the maximum number of daily flights between city i and city j. The optimal solution to this maximum flow problem is $z = x_0 = 3$, $x_{J,S} = 3$, $x_{S,L} = 1$, $x_{S,De} = 2$, $x_{L,D} = 1$, $x_{De,D} = 2$. Thus, Fly-by-Night can send three flights daily connecting Juneau and Dallas. One flight connects via Juneau–Seattle–L.A.–Dallas, and two flights connect via Juneau–Seattle–Denver–Dallas.