Math 323 QUIZ 4 Name: Sample Solution

1. A batch of four jobs J1, J2, J3, J4 can be assigned to five different machines. The set-up time for each job on various machines M1, M2, M3, M4, M5 is given the following table.
Find an optimal assignment of jobs to machines which will minimize the total time.

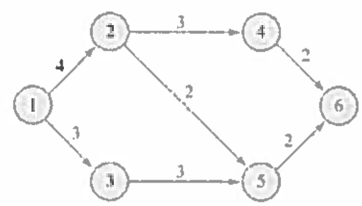
	M1	M2	M3	M4	M5
J1	10	11	4	2	8
J2	7	11	10	14	12
J3	5	6	9	12	14
J4	13	15	11	10	7
J5	0	0	$\boxed{0}$	0	0

$$\rightarrow \begin{bmatrix} 8 & 9 & 2 & 0 & 6 \\ 0 & 4 & 3 & 7 & 5 \\ 0 & 1 & 4 & 7 & 9 \\ 6 & 8 & 4 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 9 & 9 & 2 & \boxed{0} & 7 \\ \boxed{0} & 3 & 2 & 6 & 5 \\ 0 & \boxed{0} & 3 & 6 & 9 \\ 6 & 7 & 3 & 2 & \boxed{0} \\ 0 & 0 & \boxed{0} & 0 & 1 \end{bmatrix}$$

∴ This is the optimal assignment.

Time = 7 + 7 + 6 + 2 = 22

2. Find the short path from node 1 to node 6 in the following network.



$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array}$$

$[0^* \ 4 \ 3 \ \infty \ \infty \ \infty]$

$[0^* \ 4 \ 3^* \ \infty \ \infty \ \infty]$

$[0^* \ 4^* \ 3^* \ \infty \ 6 \ \infty]$

$[0^* \ 4^* \ 3^* \ 7 \ 6^* \ \infty]$

$[0^* \ 4^* \ 3^* \ 7^* \ 6^* \ 8]$

$[0^* \ 4^* \ 3^* \ 7^* \ 6^* \ 8^*]$

∴ $1 \to 2 \to 5 \to 6$ length 8

is the shortest path

Crashing the project

Suppose that by allocating additional resources to an activity, Widgetco can reduce the duration of any activity by as many as 5 days. The cost per day of reducing the duration of an activity is shown in Table 14. To find the minimum cost of completing the project by the 25-day deadline, define variables A, B, C, D, E, and F as follows:

A = number of days by which duration of activity A is reduced

⋮ ⋮

F = number of days by which duration of activity F is reduced

x_j = time that the event corresponding to node j occurs

TABLE 14

A	B	C	D	E	F
$10	$20	$3	$30	$40	$50

Then Widgetco should solve the following LP:

$0x_1 + 0x_2 + \cdots + 0x_6$

$$\min z = \boxed{10A + 20B + 3C + 30D + 40E + 50F}$$

$$\begin{array}{llll}
\text{s.t.} & A \le 5 & x_2 \ge x_1 + 9 - B & \text{(Arc (1, 2) constraint)} \\
& B \le 5 & x_3 \ge x_1 + 6 - A & \text{(Arc (1, 3) constraint)} \\
& C \le 5 & x_5 \ge x_3 + 8 - C & \text{(Arc (3, 5) constraint)} \\
& D \le 5 & x_4 \ge x_3 + 7 - D & \text{(Arc (3, 4) constraint)} \\
& E \le 5 & x_5 \ge x_4 + 10 - E & \text{(Arc (4, 5) constraint)} \\
& F \le 5 & x_6 \ge x_5 + 12 - F & \text{(Arc (5, 6) constraint)} \\
& & x_3 \ge x_2 + 0 & \text{(Arc (2, 3) constraint)} \\
& & x_6 - x_1 \le 25 & \\
& & A, B, C, D, E, F \ge 0, x_j \text{urs} &
\end{array}$$

The first six constraints stipulate that the duration of each activity can be reduced by at most 5 days. As before, the next seven constraints ensure that event j cannot occur until after node i occurs and activity (i, j) is completed. For example, activity B (arc (1, 2)) now has a duration of $9 - B$. Thus, we need the constraint $x_2 \ge x_1 + (9 - B)$. The constraint $x_6 - x_1 \le 25$ ensures that the project is completed within the 25-day deadline. The objective function is the total cost incurred in reducing the duration of the activities. An optimal solution to this LP is $z = \$390$, $x_1 = 0$, $x_2 = 4$, $x_3 = 4$, $x_4 = 6$, $x_5 = 13$, $x_6 = 25$, $A = 2$, $B = 5$, $C = 0$, $D = 5$, $E = 3$, $F = 0$. After reducing the durations of projects B, A, D, and E by the given amounts, we obtain the project network pictured in Figure 35. The reader should verify that A, B, D, E, and F are critical activities and that 1–2–3–4–5–6 and 1–3–4–5–6 are both critical paths (each having length 25). Thus, the project deadline of 25 days can be met for a cost of $390.

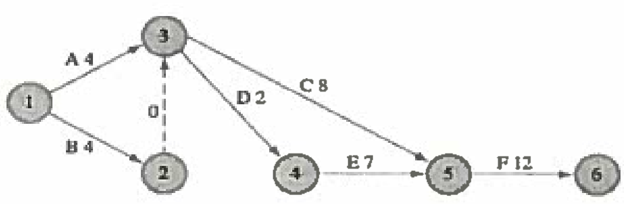


FIGURE 35
Duration of Activities after Crashing

Matlab code

```
c = [0 0 0 0 0 0 10 20 3 30 40 50];
A = [1 -1 0 0 0 0 0 -1 0 0 0 0; 1 0 -1 0 0 0 -1 0 0 0 0 0;
     0 0 1 0 -1 0 0 0 -1 0 0 0; 0 0 1 -1 0 0 0 0 0 -1 0 0;
     0 0 0 1 -1 0 0 0 0 0 -1 0; 0 0 0 0 1 -1 0 0 0 0 0 -1;
     0 1 -1 0 0 0 0 0 0 0 0 0; -1 0 0 0 0 1 0 0 0 0 0 0];
b = [-9 -6 -8 -7 -10 -12 0 25];
AA = [1 0 0 0 0 0 0 0 0 0 0 0];
bb = [0];
LB = [-20 -20 -20 -20 -20 -20 0 0 0 0 0 0];
UB = [100 100 100 100 100 100 5 5 5 5 5 5];
[x,fval] = linprog(c,A,b,AA,bb,LB,UB)
Optimal solution found.
   x^T = [0  4  4  6  13  25  2  5  0  5  3  0]
   fval = 390
```

PERT: Program Evaluation and Review Technique

- CPM assumes that the duration of each activity is known with certainty.
- For many projects, this is clearly not applicable.
- PERT is an attempt to correct this shortcoming of CPM by modeling the duration of each activity as a random variable.
- For each activity, PERT requires that the estimate the following three quantities:

  a = estimate of the activitys duration under the most favorable conditions

  b = estimate of the activitys duration under the least favorable conditions

  m = most likely value for the activitys duration

- Let $\mathbf{T}_{ij}$ (random variables are printed in boldface) be the duration of activity (i, j).
- PERT requires the assumption that $\mathbf{T}_{ij}$ follows a beta distribution.
- The specific definition of a beta distribution need not concern us, but it is important to realize that it can approximate a wide range of random variables, including many positively skewed, negatively skewed, and symmetric random variables.
- If $\mathbf{T}_{ij}$ follows a beta distribution, then it can be shown that the mean and variance of $\mathbf{T}_{ij}$ may be approximated by

$$\boxed{E(\mathbf{T}_{ij}) = \frac{a + 4m + b}{6}} \quad \text{and} \quad \boxed{var(\mathbf{T}_{ij}) = \frac{(b-a)^2}{36}.}$$

- PERT requires the assumption that the durations of all activities are independent.
- Then for any path in the project network, the mean and variance of the time required to complete the activities on any path P are given by

$$\boxed{\sum_{(i,j)\in P} E(\mathbf{T}_{ij})} \quad \text{and} \quad \boxed{\sum_{(i,j)\in P} var(\mathbf{T}_{ij}).}$$

- Let CP be the random variable denoting the total duration of the activities on a critical path found by CPM.
- PERT assumes that the critical path $\hat{P}$ found by CPM contains enough activities to allow us to invoke the Central Limit Theorem and conclude that

$$\boxed{CP = \sum_{(i,j)\in \hat{P}} \mathbf{T}_{ij}}$$

  is normally distributed.

- Then one can answer questions concerning the probability that the project will be completed by a given date.

Example Consider

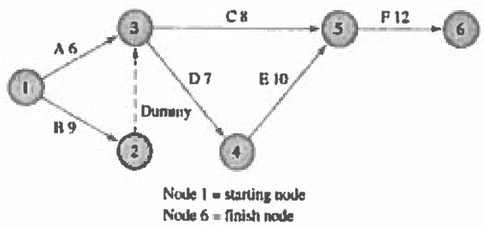


TABLE 15
a, b, and m for Activities in Widgets

Activity	a	b	m
(1, 2)	5	13	9
(1, 3)	2	10	6
(3, 5)	3	13	8
(3, 4)	1	13	7
(4, 5)	8	12	10
(5, 6)	9	15	12

Then we have

$$E(T_{12}) = \frac{\{5 + 13 + 36\}}{6} = 9 \qquad \mathrm{var}T_{12} = \frac{(13-5)^2}{36} = 1.78$$

$$E(T_{13}) = \frac{\{2 + 10 + 24\}}{6} = 6 \qquad \mathrm{var}T_{13} = \frac{(10-2)^2}{36} = 1.78$$

$$E(T_{35}) = \frac{\{3 + 13 + 32\}}{6} = 8 \qquad \mathrm{var}T_{35} = \frac{(13-3)^2}{36} = 2.78$$

$$E(T_{34}) = \frac{\{1 + 13 + 28\}}{6} = 7 \qquad \mathrm{var}T_{34} = \frac{(13-1)^2}{36} = 4$$

$$E(T_{45}) = \frac{\{8 + 12 + 40\}}{6} = 10 \qquad \mathrm{var}T_{45} = \frac{(12-8)^2}{36} = 0.44$$

$$E(T_{56}) = \frac{\{9 + 15 + 48\}}{6} = 12 \qquad \mathrm{var}T_{56} = \frac{(15-9)^2}{36} = 1$$

Of course, the fact that arc (2, 3) is a dummy arc yields

$$E(T_{23}) = \mathrm{var}\, T_{23} = 0$$

Recall that the critical path for Example 6 was 1–2–3–4–5–6. From Equations (6) and (7),

$$E(\mathbf{CP}) = 9 + 0 + 7 + 10 + 12 = 38$$
$$\mathrm{var}\mathbf{CP} = 1.78 + 0 + 4 + 0.44 + 1 = 7.22$$

Then the standard deviation for **CP** is $(7.22)^{1/2} = 2.69$.

Question What is the probability that the project will be completed within 35 days?

Answer Assume that 1-2-3-4-5-6 is always the critical path. Then project will be completed within 35 days is just Prob($CP \leq 35$), and assume that CP is normally distributed. Then by the transformation $Z = (CP - 38)/2.69$ so that Z is a standardized normalized distributed random variable with mean 0 and standard deviation 1, we have

$$\mathrm{Prob}(CP \leq 35) = \mathrm{Prob}((CP - 38)/2.69 \leq (35 - 38)/2.69) = \mathrm{Prob}(Z \leq -1.12) = 0.13$$

by the standard normal distribution table.

Difficulties with PERT

1. The assumption that the activity durations are independent is difficult to justify.
2. Activity durations may not follow a beta distribution.
3. The assumption that the critical path found by CPM will always be the critical path for the project may not be justified.

- The last difficulty is the most serious.
- For instance, in our example, we assumed that 1-2-3-4-5-6 would always be the critical path.
- If, however, activity A were significantly delayed and activity B were completed ahead of schedule, then the critical path might be 1-3-4-5-6.
- One needs probability, simulation techniques to deal with the problem.