**Problem** We want to build the minimum number of connection/roads to keep a network connected. We assume that the arcs are undirected. (Network with directed arcs will be slightly more tricky.)

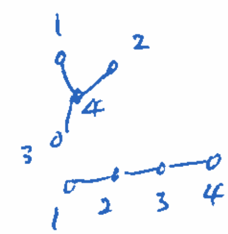
## Minimum Spanning Tree Problems

Facts: ① n vertices, need at least n-1 edges (undirected arcs / two way arcs) to connect the n vertices.

② If you have more than n-1 edges, there will be a cycle / closed walk.

③ A connected graph with n vertices and n-1 edges is called a tree.

Example




The following method (MST algorithm) may be used to find a minimum spanning tree for a network:

**Step 1** Begin at any node $i$, and join node $i$ to the node in the network (node $j$) that is closest to node $i$. The two nodes $i$ and $j$ now form a connected set of nodes $C = \{i, j\}$ and arc $(i, j)$ will be in the minimum spanning tree. The remaining nodes in the network ($C'$) are the unconnected set of nodes.

**Step 2** Choose a member of $C'(n)$ that is closest to some node in $C$. Let $m$ represent the node in $C$ that is closest to $n$. Then the arc $(m, n)$ will be in the minimum spanning tree. Update $C$ and $C'$. Because $n$ is now connected to $\{i, j\}$, $C$ now equals $\{i, j, n\}$, and we must eliminate node $n$ from $C'$.

**Step 3** Repeat this process until a minimum spanning tree is found. Ties for closest node and arc may be broken arbitrarily.

**Example**

Alternatively,

$1 + 2 + 2 + 4 = 9$

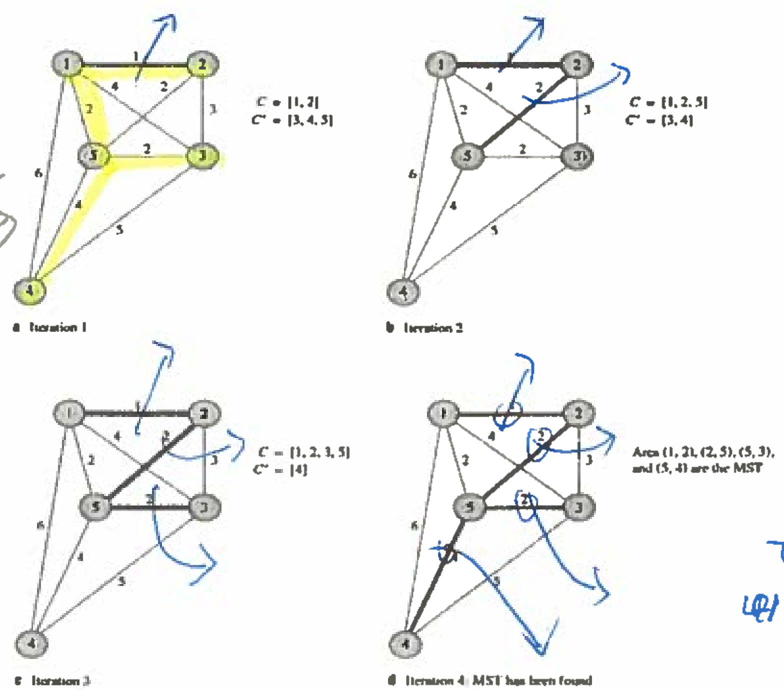
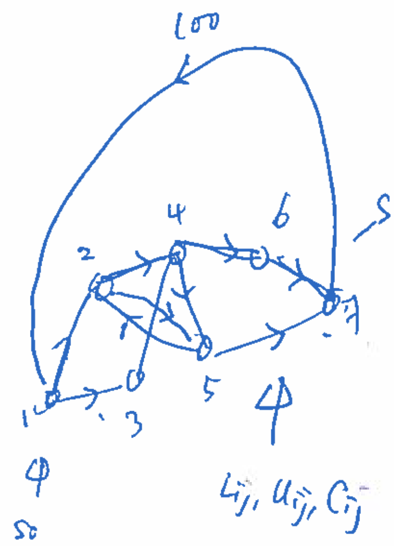



**FIGURE 49**
**MST Algorithm for Computer Example**

Total cost:

~~4~~1 + 2 + 2 + 4 = 9

## Minimum-Cost Network Flow Problems




The transportation, assignment, transshipment, shortest-path, maximum-flow, and critical path problems are all special cases of the minimum-cost network flow problem (MCNFP).

$x_{ij}$ = number of units of flow sent from node $i$ to node $j$ through arc $(i, j)$

$b_i$ = net supply (outflow − inflow) at node $i$

$c_{ij}$ = cost of transporting one unit of flow from node $i$ to node $j$ via arc $(i, j)$

$L_{ij}$ = lower bound on flow through arc $(i, j)$ (if there is no lower bound, let $L_{ij} = 0$)

$U_{ij}$ = upper bound on flow through arc $(i, j)$ (if there is no upper bound, let $U_{ij} = \infty$)

Then an MCNFP may be written as

$$\min \sum_{\text{all arcs}} c_{ij}x_{ij}$$

$$\text{s.t.} \quad \sum_j x_{ij} - \sum_k x_{ki} = b_i \qquad \text{(for each node } i \text{ in the network)}$$

$$L_{ij} \le x_{ij} \le U_{ij} \qquad \text{(for each arc in the network)}$$

The first set of constraints are the **flow balance equations,** and the second set of constraints express limitations on arc capacities.

### Transportation problems as MCNF problems

**TABLE 28**

| | Node 3 | Node 4 | Supply |
|---|---|---|---|
| Node 1 | 1 | 2 | 4 (Node 1) |
| Node 2 | 3 | 4 | 5 (Node 2) |
| Demand | 6 (Node 3) | 3 (Node 4) | |

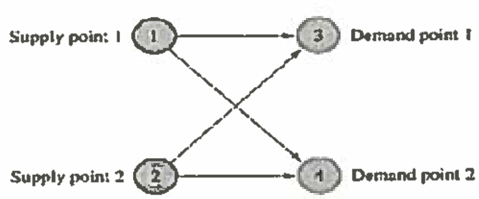


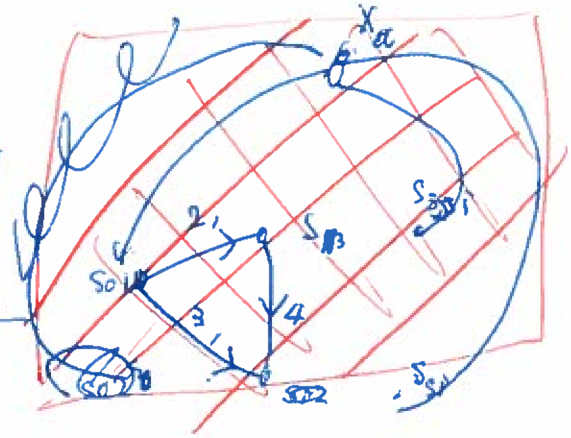

**FIGURE 45**
**Representation of Transportation Problem as an MCNFP**

**TABLE 29**
**MCNFP Representation of Transportation Problem**

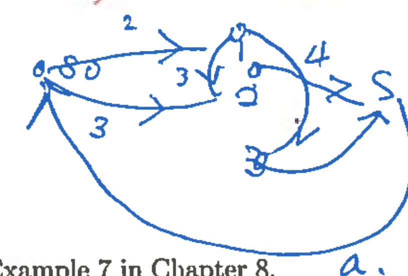
| $\min z = x_{13} + 2x_{14} + 3x_{23} + 4x_{24}$ | | | | | | |
|---|---|---|---|---|---|---|
| $x_{13}$ | $x_{14}$ | $x_{23}$ | $x_{24}$ | | rhs | Constraint |
| 1 | 1 | 0 | 0 | = | 4 | Node 1 |
| 0 | 0 | 1 | 1 | = | 5 | Node 2 |
| −1 | 0 | −1 | 0 | = | −6 | Node 3 |
| 0 | −1 | 0 | −1 | = | −3 | Node 4 |
| All variables non-negative | | | | | | |

## Maximum flow problems as MCNF problems

**TABLE 30**
**MCNFP Representation of Maximum-Flow Problem**



| max $z = x_0$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $x_{so,1}$ | $x_{so,2}$ | $x_{13}$ | $x_{12}$ | $x_{3,si}$ | $x_{2,si}$ | $x_0$ | | rhs | Constraint |
| 1 | 1 | 0 | 0 | 0 | 0 | −1 | = | 0 | Node *so* |
| −1 | 0 | 1 | 1 | 0 | 0 | 0 | = | 0 | Node 1 |
| 0 | −1 | 0 | −1 | 0 | 1 | 0 | = | 0 | Node 2 |
| 0 | 0 | −1 | 0 | 1 | 0 | 0 | = | 0 | Node 3 |
| 0 | 0 | 0 | 0 | −1 | −1 | 1 | = | 0 | Node *si* |
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | ≤ | 2 | Arc (*so*, 1) |
| 0 | 1 | 0 | 0 | 0 | 0 | 0 | ≤ | 3 | Arc (*so*, 2) |
| 0 | 0 | 1 | 0 | 0 | 0 | 0 | ≤ | 4 | Arc (1, 3) |
| 0 | 0 | 0 | 1 | 0 | 0 | 0 | ≤ | 3 | Arc (1, 2) |
| 0 | 0 | 0 | 0 | 1 | 0 | 0 | ≤ | 1 | Arc (3, *si*) |
| 0 | 0 | 0 | 0 | 0 | 1 | 0 | ≤ | 2 | Arc (2, *si*) |
| All variables nonnegative | | | | | | | | | |



### Remarks

- One can formulate other network problems as MCNF problems; see Example 7 in Chapter 8.
- One can apply a change of variables $x_{ij} - L_{ij}$ and assume that $L_{ij} = 0$. Note that $b_i$ will be changed to

$$\tilde{b}_i = b_i - \sum_j L_{ij} + \sum_k L_{ki} = \sum_j (x_{ij} - L_{ij}) - \sum_k (x_{ki} - L_{ki}).$$

- One can use linprog(c,A,b,AA,bb,LB,UB) command in Matlab to solve the network problem by setting up

```
the cost vector c = [c_{ij}],  A = [ ], b = [ ],
AA x = bb, the network constraints,  LB, UB.
```

- Note that we can delete one of network constraints $\tilde{A}x = \tilde{b}$ to get $AAx = bb$ because each column of $\tilde{A}$ has a "1" and a "−1".

  So, the sum of rows of $\tilde{A} = [0, \ldots, 0]$, i.e., the rows are linearly dependent.

Because the tableau $\left[\begin{array}{c|c} A & b \\ \hline c & \end{array}\right]$ corresponding to a network model has simple structure, namely, every column of $A$ has only 2 non-zero entries, 1 and $-1$.

## Network Simplex Method

**Step 1** Determine a starting bfs. The $n - 1$ basic variables will correspond to a spanning tree. Indicate nonbasic variables at their upper bound by dashed arcs.

**Step 2** Compute $y_1, y_2, \ldots y_n$ (often called the *simplex multipliers*) by solving $y_1 = 0$, $y_i - y_j = c_{ij}$ for all basic variables $x_{ij}$. For all nonbasic variables, determine the row 0 coefficient $\bar{c}_{ij}$ from $\bar{c}_{ij} = y_i - y_j - c_{ij}$. The current bfs is optimal if $\bar{c}_{ij} \leq 0$ for all $x_{ij} = L_{ij}$ and $\bar{c}_{ij} \geq 0$ for all $x_{ij} = U_{ij}$. If the bfs is not optimal, then choose the nonbasic variable that most violates the optimality conditions as the entering basic variable.

**Step 3** Identify the cycle (there will be exactly one!) created by adding the arc corresponding to the entering variable to the current spanning tree of the current bfs. Use conservation of flow to determine the new values of the variables in the cycle. The variable that first hits its upper or lower bound as the value of the entering basic variable is changed exits the basis.

**Step 4** Find the new bfs by changing the flows of the arcs in the cycle found in step 3. Go to step 2.

We can use special algorithm to solve such problems

Remark. For a connected directed network:

How to set up the min cost network model?

Question: If we split the arc into ⇅ How to assign cost

## EXAMPLE 9 Network Simplex Solution to MCNFP

Use the network simplex to solve the MCNFP in Figure 56.

**Solution** A bfs requires that we find a spanning tree (three arcs that connect nodes 1, 2, 3, and 4 and do not form a cycle). Any arcs not in the spanning tree may be set equal to their upper or lower bound. By trial and error, we find the bfs in Figure 57 involving the spanning tree (1, 2), (1, 3), and (2, 4).

To find $y_1, y_2, y_3$, and $y_4$ we solve

$$y_1 = 0, \qquad y_1 - y_2 = 4, \qquad y_2 - y_4 = 3, \qquad y_1 - y_3 = 3$$

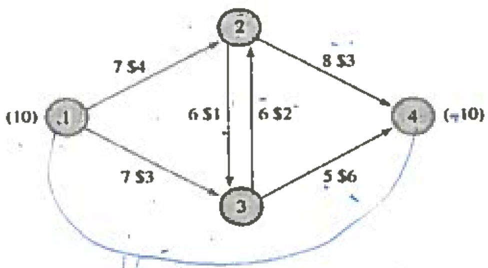



**FIGURE 56**
**Example of Network Simplex**

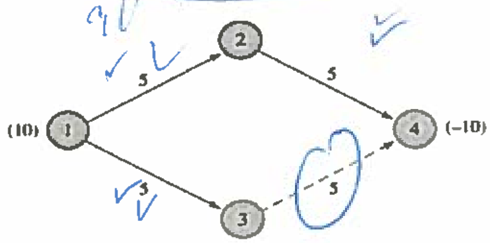




**FIGURE 57**
**bfs for Example 9**

This yields $y_1 = 0$, $y_2 = -4$, $y_3 = -3$, and $y_4 = -7$. The row 0 coefficients for each nonbasic variable are

$\bar{c}_{34} = -3 - (-7) - 6 = -2$ (Violates optimality condition)

$\bar{c}_{23} = -4 - (-3) - 1 = -2$ (Satisfies optimality condition)

$\bar{c}_{32} = -3 - (-4) - 2 = -1$ (Satisfies optimality condition)

Thus, $x_{34}$ enters the basis. We set $x_{34} = 5 - \theta$ and obtain the cycle in Figure 58. From arc (1, 2), we find $5 + \theta \le 7$ or $\theta \le 2$. From arc (1, 3), we find $5 - \theta \ge 0$ or $\theta \le 5$. From arc (2, 4), we find $5 + \theta \le 8$ or $\theta \le 3$. From arc (3, 4), we find $5 - \theta \ge 0$ or $\theta \le 5$. Thus, we can set $\theta = 2$. Now $x_{12}$ exits the basis at its upper bound, and $x_{34}$ enters, yielding the bfs in Figure 59.

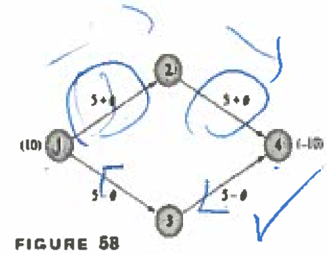



FIGURE 58
Cycle Created When $x_{34}$ Enters the Basis

$\theta = 2$

The new bfs is associated with the spanning tree (1, 3), (2, 4), and (3, 4). Solving for the new values of the simplex multipliers, we obtain

$$y_1 = 0, \quad y_1 - y_3 = 3, \quad y_3 - y_4 = 6, \quad y_2 - y_4 = 3$$

This yields $y_1 = 0, y_2 = -6, y_3 = -3, y_4 = -9$. The coefficient of each nonbasic variable in row 0 is given by

$\bar{c}_{12} = 0 - (-6) - 4 = 2$ (Satisfies optimality condition)

$\bar{c}_{23} = -6 - (-3) - 1 = -4$ (Satisfies optimality condition)

$\bar{c}_{32} = -3 - (-6) - 2 = 1$ (Violates optimality condition)

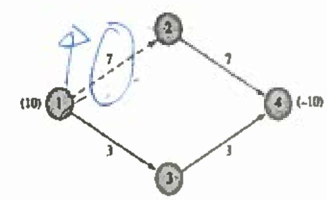
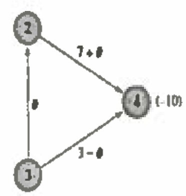



FIGURE 59
bfs After $x_{12}$ Exits and $x_{34}$ Enters

Now $x_{32}$ enters the basis, yielding the cycle in Figure 60. From arc (2, 4), we find $7 + \theta \le 8$ or $\theta \le 1$); from arc (3, 4), we find $3 - \theta \ge 0$ or $\theta \le 3$. From arc (3, 2), we find $\theta \le 6$. So we now set $\theta = 1$ and have $x_{24}$ exit from the basis at its upper bound. The new bfs is given in Figure 61.

$\theta = 1$

FIGURE 60
Cycle Created When $x_{32}$ Enters Basis




The current set of basic values corresponds to the spanning tree (1, 3), (3, 2), and (3, 4). The new values of the simplex multipliers are found by solving

$$y_1 = 0, \quad y_1 - y_3 = 3, \quad y_3 - y_2 = 2, \quad y_3 - y_4 = 6$$

which yields $y_1 = 0, y_2 = -5, y_3 = -3, y_4 = -9$. The coefficient of each nonbasic variable in row 0 is now

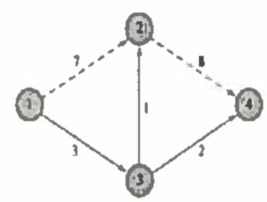



FIGURE 61
New bfs When $x_{32}$ Enters and $x_{24}$ Exits

$\bar{c}_{23} = -5 - (-3) - 1 = -3$ (Satisfies optimality condition)

$\bar{c}_{12} = 0 - (-5) - 4 = 1$ (Satisfies optimality condition)

$\bar{c}_{24} = -5 - (-9) - 3 = 1$ (Satisfies optimality condition)

Thus, the current bfs is optimal. The optimal solution to the MCNFP is

Basic variables: $x_{13} = 3$, $x_{32} = 1$, $x_{34} = 2$

Nonbasic variables at their upper bound: $x_{12} = 7$, $x_{24} = 8$

Nonbasic variable at lower bound: $x_{23} = 0$

The optimal $z$-value is obtained from

$$z = 7(4) + 3(3) + 1(2) + 8(3) + 2(6) = \$75$$