§9.1 Integer programming

- An IP in which all variables are required to be integers is called a pure integer programming problem.
- An IP in which only some of the variables are required to be integers is called a mixed integer programming problem.
- An integer programming problem in which all the variables must equal 0 or 1 is called a 0–1 IP.
- The LP obtained by omitting all integer or 0–1 constraints on variables is called the LP relaxation of the IP.
- the LP relaxation is a less constrained, or more relaxed, version of the IP.
- This means that the feasible region for any IP must be contained in the feasible region for the corresponding LP relaxation.
- So, the optimal value for the LP relaxation is better than that of the original problem.

Example Consider the following simple IP:

$$\max Z = 21x_1 + 11x_2$$

$$\text{subjecct to} \quad 7x_1 + 4x_2 \le 13$$

$$x_1, x_2 \ge 0; x_1, x_2 \text{ integer}$$

From Figure 1, we see that the feasible region for this problem consists of the following set of points $S = \{(0,0),(0,1),(0,2),(0,3),(1,0),(1,1)\}$.

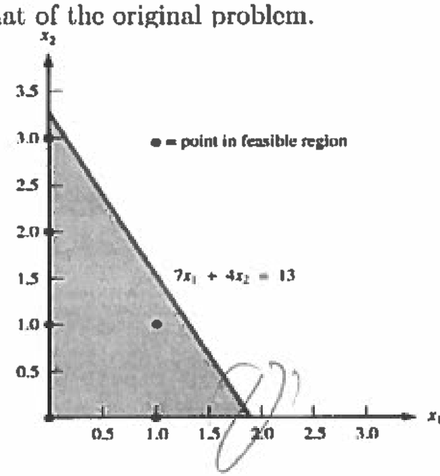

FIGURE 1
Feasible Region for Simple IP (4)

- One may solve the LP relaxation, and check all the integer points to determine the optimal.
- In our example, we find the optimal solution is $Z = 33$ with $(x_1, x_2) = (0, 3)$.
- However, it is not practical as there may be billions of integer points in the feasible regions.
- Another simple idea is to solve the LP relaxation; then round off the variables to the nearest integer for each variable.
- For our problem, the LP relaxation has optimal solution: $(x_1, x_2) = (13/7, 0)$. Rounding this solution up yields the solution $(x_1, x_2) = (2, 0)$, which is infeasible. Rounding the solution down yields the solution $(x_1, x_2) = (1, 0)$, which is not optimal.
- We need new techniques.

§9.2 Formulations

Example 1 Stockco consider 4 investments:

Investment	Net present value (NPV)	Cash outflow
1	16000	5000
2	22000	7000
3	12000	4000
4	8000	3000

Now, \$14000 is available. How to invest?

Formulation. $\max Z = 16x_1 + 22x_2 + 12x_3 + 8x_4$

subject to $5x_1 + 7x_2 + 4x_3 + 3x_4 \leq 14$.

Set $x_i = 1$ or 0 for $i = 1, 2, 3, 4$.

Exmple 2 Modify the Stockco formulation to account for each of the following requirements:

1. Stockco can invest in at most two investments.
2. If Stockco invests in investment 2, they must also invest in investment 1.
3. If Stockco invests in investment 2, they cannot invest in investment 4.

Modification

1. Simply add the constraint $x_1 + x_2 + x_3 + x_4 \leq 2$.
2. We add the constraint $x_2 \leq x_1$ or $x_2 - x_1 \leq 0$.
3. We add the constraint $x_2 + x_4 \leq 1$.

Example 3 Fixed charge IP. Gandhi Cloth Company is capable of manufacturing three types of clothing: shirts, shorts, and pants.

- The manufacture of each type of clothing requires that Gandhi have the appropriate type of machinery available.
- The machinery needed to manufacture each type of clothing must be rented at the following rates: shirt machinery, \$200 per week; shorts machinery, \$150 per week; pants machinery, \$100 per week.
- The manufacture of each type of clothing also requires the amounts of cloth and labor shown in Table 2.
- Each week, 150 hours of labor and 160 sq yd of cloth are available.
- The variable unit cost and selling price for each type of clothing are shown in Table 3.

TABLE 2
Resource Requirements for Gandhi

Clothing Type	Labor (Hours)	Cloth (Square Yards)
Shirt	3	4
Shorts	2	3
Pants	6	4

TABLE 3
Revenue and Cost Information for Gandhi

Clothing Type	Sales Price (\$)	Variable Cost (\$)	
Shirt	12	6	=6
Shorts	8	4	=4
Pants	15	8	

Formulate an IP whose solution will maximize Gandhis weekly profits.

Formulation Let x_1, x_2, x_3 be the number of shirts, shorts, and pants, produced each week;

$$y_1 = \begin{cases} 1 & \text{if any shirts are manufactured} \\ 0 & \text{otherwise} \end{cases}$$

$$y_2 = \begin{cases} 1 & \text{if any shorts are manufactured} \\ 0 & \text{otherwise} \end{cases}$$

$$y_3 = \begin{cases} 1 & \text{if any pants are manufactured} \\ 0 & \text{otherwise} \end{cases}$$

In short, if $x_j > 0$, then $y_j = 1$, and if $x_j = 0$, then $y_j = 0$. Thus, Gandhi's weekly profits = (weekly sales revenue) − (weekly variable costs) − (weekly costs of renting machinery).

We can then formulate the problem as:

ϕ 0,1 ϕ 0,1 ϕ 0,1   8 choice

$$\begin{aligned} \max z = {} & 6x_1 + 4x_2 + 7x_3 \boxed{- 200y_1 - 150y_2 - 100y_3} \\ \text{s.t.} \quad & 3x_1 + 2x_2 + 6x_3 \le 150 \\ & 4x_1 + 3x_2 + 4x_3 \le 160 \\ & x_1 \le M_1 y_1 \\ & x_2 \le M_2 y_2 \\ & x_3 \le M_3 y_3 \\ & x_1, x_2, x_3 \ge 0;\ x_1, x_2, x_3 \text{ integer} \\ & y_1, y_2, y_3 = 0 \text{ or } 1 \end{aligned}$$

The optimal solution to the Gandhi problem is $z = \$75$, $x_3 = 25, y_3 = 1$. Thus, Gandhi should produce 25 pants each week.

Example 4 The Lockbox problem J. C. Nickles receives credit card payments from four regions of the country (West, Midwest, East, and South).

The average daily value of payments mailed by customers from each region is as follows:

the West, \$70,000; the Midwest, \$50,000; the East, \$60,000; the South, \$40,000.

Nickles must decide where customers should mail their payments.

Because Nickles can earn 20% annual interest by investing these revenues, it would like to receive payments as quickly as possible.

Nickles is considering setting up operations to process payments (often referred to as lockboxes) in four different cities:

Los Angeles, Chicago, New York, and Atlanta.

The average number of days (from time payment is sent) until a check clears and Nickles can deposit the money depends on the city to which the payment is mailed, as shown in Table 4.

TABLE 4

Average Number of Days from Mailing of Payment Until Payment Clears

From	To: City 1 (Los Angeles)	City 2 (Chicago)	City 3 (New York)	City 4 (Atlanta)
Region 1 West	2	6	8	8
Region 2 Midwest	6	2	5	5
Region 3 East	8	5	2	5
Region 4 South	8	5	5	2

For example, if a check is mailed from the West to Atlanta, it would take an average of 8 days before Nickles could earn interest on the check.

The annual cost of running a lockbox in any city is \$50,000.

Formulate an IP that Nickles can use to minimize the sum of costs due to lost interest and lockbox operations.

Let $x_{ij} \in \{0,1\}$ so that 1 means regions i send checks to region j, and let $y_j \in \{0,1\}$ so that 1 means that mailbox is operated at city j.

$$\begin{aligned}
\min z = {} & 28x_{11} + 84x_{12} + 112x_{13} + 112x_{14} + 60x_{21} + 20x_{22} + 50x_{23} + 50x_{24} \\
& + 96x_{31} + 60x_{32} + 24x_{33} + 60x_{34} + 64x_{41} + 40x_{42} + 40x_{43} + 16x_{44} \\
& + 50y_1 + 50y_2 + 50y_3 + 50y_4
\end{aligned}$$

$$\begin{aligned}
\text{s.t.}\quad & x_{11} + x_{12} + x_{13} + x_{14} = 1 && \text{(West region constraint)} \\
& x_{21} + x_{22} + x_{23} + x_{24} = 1 && \text{(Midwest region constraint)} \\
& x_{31} + x_{32} + x_{33} + x_{34} = 1 && \text{(East region constraint)} \\
& x_{41} + x_{42} + x_{43} + x_{44} = 1 && \text{(South region constraint)}
\end{aligned}$$

$$x_{11} \le y_1, x_{21} \le y_1, x_{31} \le y_1, x_{41} \le y_1, x_{12} \le y_2, x_{22} \le y_2, x_{32} \le y_2, x_{42} \le y_2,$$
$$x_{13} \le y_3, x_{23} \le y_3, x_{33} \le y_3, x_{43} \le y_3, x_{14} \le y_4, x_{24} \le y_4, x_{34} \le y_4, x_{44} \le y_4$$

$$\text{All } x_{ij} \text{ and } y_j = 0 \text{ or } 1$$

The optimal solution is $z = 242$, $y_1 = 1$, $y_3 = 1$, $x_{11} = 1$, $x_{23} = 1$, $x_{33} = 1$, $x_{43} = 1$. Thus, Nickles should have a lockbox operation in Los Angeles and New York. West customers should send payments to Los Angeles, and all other customers should send payments to New York.

There is an alternative way of modeling the Type 2 constraints. Instead of the 16 constraints of the form $x_{ij} \le y_j$, we may include the following four constraints:

$$\begin{aligned}
& x_{11} + x_{21} + x_{31} + x_{41} \le 4y_1 && \text{(Los Angeles constraint)} \\
& x_{12} + x_{22} + x_{32} + x_{42} \le 4y_2 && \text{(Chicago constraint)} \\
& x_{13} + x_{23} + x_{33} + x_{43} \le 4y_3 && \text{(New York constraint)} \\
& x_{14} + x_{24} + x_{34} + x_{44} \le 4y_4 && \text{(Atlanta constraint)}
\end{aligned}$$

TABLE 5

Calculation of Annual Lost Interest

Assignment	Annual Lost Interest Cost (\$)
West to L.A.	0.20(70,000)2 = 28,000
West to Chicago	0.20(70,000)6 = 84,000
West to N.Y.	0.20(70,000)8 = 112,000
West to Atlanta	0.20(70,000)8 = 112,000
Midwest to L.A.	0.20(50,000)6 = 60,000
Midwest to Chicago	0.20(50,000)2 = 20,000
Midwest to N.Y.	0.20(50,000)5 = 50,000
Midwest to Atlanta	0.20(50,000)5 = 50,000
East to L.A.	0.20(60,000)8 = 96,000
East to Chicago	0.20(60,000)5 = 60,000
East to N.Y.	0.20(60,000)2 = 24,000
East to Atlanta	0.20(60,000)5 = 60,000
South to L.A.	0.20(40,000)8 = 64,000
South to Chicago	0.20(40,000)5 = 40,000
South to N.Y	0.20(40,000)5 = 40,000
South to Atlanta	0.20(40,000)2 = 16,000

Think about

C_1 C_5 C_3 C_2 C_4 C_6

Find disks 15 min to cover all these cities.

Example 5 set covering problem There are six cities (cities 16) in Kilroy County. The county must determine where to build fire stations. The county wants to build the minimum number of fire stations needed to ensure that at least one fire station is within 15 minutes (driving time) of each city.

The times (in minutes) required to drive between the cities in Kilroy County are shown in Table 6. Formulate an IP that will tell Kilroy how many fire stations should be built and where they should be located.

TABLE 6
Time Required to Travel between Cities in Kilroy County

From	To					
	City 1	City 2	City 3	City 4	City 5	City 6
City 1	0	10	20	30	30	20
City 2	10	0	25	35	20	10
City 3	20	25	0	15	30	20
City 4	30	35	15	0	15	25
City 5	30	20	30	15	0	14
City 6	20	10	20	25	14	0

Formulation Let $x_i \in \{0,1\}$ so that $x_i = 1$ means that a fire station is built in city i.

TABLE 7
Cities within 15 Minutes of Given City

City	Within 15 Minutes
1	1, 2
2	1, 2, 6
3	3, 4
4	3, 4, 5
5	4, 5, 6
6	2, 5, 6

$$\boxed{\min z = x_1 + x_2 + x_3 + x_4 + x_5 + x_6}$$

$$\begin{array}{llr}
\text{s.t.} & x_1 + x_2 \geq 1 & \text{(City 1 constraint)} \\
& x_1 + x_2 + x_6 \geq 1 & \text{(City 2 constraint)} \\
& x_3 + x_4 \geq 1 & \text{(City 3 constraint)} \\
& x_3 + x_4 + x_5 \geq 1 & \text{(City 4 constraint)} \\
& x_4 + x_5 + x_6 \geq 1 & \text{(City 5 constraint)} \\
& x_2 + x_5 + x_6 \geq 1 & \text{(City 6 constraint)} \\
& x_i = 0 \text{ or } 1 \quad (i = 1, 2, 3, 4, 5, 6) &
\end{array}$$

One optimal solution to this IP is $z = 2$, $x_2 = x_4 = 1$, $x_1 = x_3 = x_5 = x_6 = 0$. Thus, Kilroy County can build two fire stations: one in city 2 and one in city 4.

Either–Or Constraints

The following situation commonly occurs in mathematical programming problems. We are given two constraints of the form

$$f(x_1, x_2, \ldots, x_n) \leq 0 \tag{26}$$

$$g(x_1, x_2, \ldots, x_n) \leq 0 \tag{27}$$

We want to ensure that at least one of (26) and (27) is satisfied, often called **either–or constraints.** Adding the two constraints (26′) and (27′) to the formulation will ensure that at least one of (26) and (27) is satisfied:

$$f(x_1, x_2, \ldots, x_n) \leq My \tag{26′}$$

$$g(x_1, x_2, \ldots, x_n) \leq M(1 - y) \tag{27′}$$