Quiz on Thursday will cover material in Homework 10.

9.3 The Branch-and-Bound Method for Solving Pure Integer Programming Problems

- Most IPs are solved by using the technique of branch-and-bound.
- Branch-and-bound methods find the optimal solution to an IP by efficiently enumerating the points in a subproblems feasible region.
- Note: If you solve the LP relaxation of a pure IP and obtain a solution in which all variables are integers, then the optimal solution to the LP relaxation is also the optimal solution to the IP.

Example 9 The Telfa Corporation manufactures tables and chairs.

- A table requires 1 hour of labor and 9 square board feet of wood.
- A chair requires 1 hour of labor and 5 square board feet of wood.
- Currently, 6 hours of labor and 45 square board feet of wood are available.
- Each table contributes \$8 to profit, and each chair contributes \$5 to profit.

Formulation of the IP to maximize Telfas profit.

Let

$$x_1 = \text{number of tables manufactured}$$
$$x_2 = \text{number of chairs manufactured}$$

Because x_1 and x_2 must be integers, Telfa wants to solve the following IP:

$$\begin{aligned}
\max z &= 8x_1 + 5x_2 \\
\text{s.t.} \quad x_1 + x_2 &\le 6 \quad \text{(Labor constraint)} \\
9x_1 + 5x_2 &\le 45 \quad \text{(Wood constraint)} \\
x_1, x_2 &\ge 0;\ x_1, x_2 \text{ integer}
\end{aligned}$$

Solving the problem

- The branch-and-bound method begins by solving the LP relaxation of the IP.
- If all the decision variables assume integer values in the optimal solution to the LP relaxation, then we are done.
- We call the LP relaxation subproblem 1.
- Here the optimal solution to the LP relaxation is $z = 165/4, x_1 = 15/4, x_2 = 9/4$ (see Figure 11).

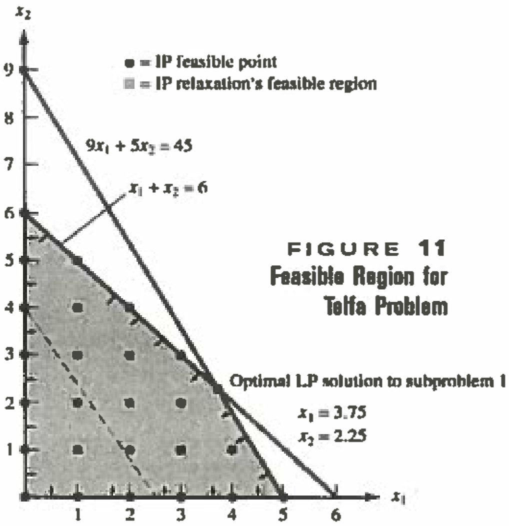


FIGURE 11
Feasible Region for Telfa Problem

- From Section 9.1, we have (optimal Z-value for IP) $\leq$ (optimal Z-value for LP relaxation).
- This implies that the optimal z-valuefor the IP cannot exceed 165/4.
- Thus, the optimal z-value for the LP relaxation is an upper bound for Telfas profit.
- We partition the feasible region for the LP relaxation in an attempt to find out more about the location of the IPs optimal solution.
- Choose a variable that is fractional in the optimal solution to the LP relaxation-say, x_1.
- Note that every point in the feasible region for the IP must have either $x_1 \leq 3$ or $x1 \geq 4$. (Why cant a feasible solution to the IP have $3 < x_1 < 4$?)
- With this in mind, we "branch" on the variable x_1 and create two additional subproblems.
- The optimal solution to subproblem 2 did not yield an all-integer solution.
- Choose a fractional valued variable x_2 in the optimal solution to subproblem 2 and then branch on that variable.
- Partition the feasible region for subproblem 2 into those points having $x_2 \geq 2$ and $x_2 \leq 1$, and get the following two subproblems:

Subproblem 2 Subproblem 1 + Constraint $x_1 \geq 4$.

Subproblem 3 Subproblem 1 + Constraint $x_1 \leq 3$.

- Neither subproblem 2 nor subproblem 3 includes any points with $x_1 = 15/4$.
- The optimal solution to the LP relaxation cannot recur when we solve subproblem 2 or subproblem 3.
- From Figure 12, every point in the feasible region for the Telfa IP is included in the feasible region for subproblem 2 or subproblem 3.
- The feasible regions for subproblems 2 and 3 have no points in common.
- We say that subproblems 2 and 3 were created by branching on x_1.
- Choose any subproblem, say, subproblem 2, that has not yet been solved as an LP.
- From Figure 12, we see that the optimal solution to subproblem 2 is $z = 41, x_1 = 4, x_2 = 9/5$ (point C). See Figure 13.
- A display of all subproblems that have been created is called a **tree**.

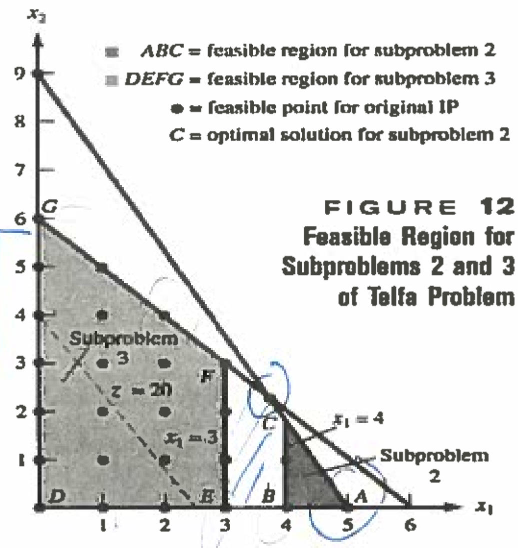


FIGURE 12
Feasible Region for Subproblems 2 and 3 of Telfa Problem

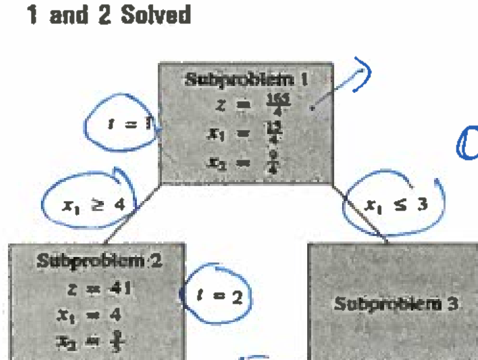


FIGURE 13
Telfa Subproblems 1 and 2 Solved

- Each subproblem is referred to as a **node** of the tree, and each line connecting two nodes of the tree is called an **arc**.
- The constraints associated with any node of the tree are the constraints for the LP relaxation plus the constraints associated with the arcs leading from subproblem 1 to the node.
- The label t indicates the chronological order in which the subproblems are solved.

Record telling you/the computer how to backtrack

Subproblem 4

Subproblem 1 + Constraints $x_1 \geq 4$ and $x_2 \geq 2$

= subproblem 2 + Constraint $x_2 \geq 2$.

Subproblem 5

Subproblem 1 + Constraints $x_1 \geq 4$ and $x_2 \leq 1$

= subproblem 2 + Constraint $x_2 \leq 1$.

- The feasible regions for subproblems 4 and 5 are displayed in Figure 14.
- The set of unsolved subproblems consists of subproblems 3, 4, and 5.
- Choose the most recently created subproblem, i.e., subproblem 4 or subproblem 5, to solve. (This is called the LIFO, or last-in-first-out, rule.) Here we choose to solve subproblem 4.
- From Figure 14 we see that subproblem 4 is infeasible. We place an $\times$ by subproblem 4 (see Figure 15).
- We say that the subproblem (or node) is fathomed. (No need to branch out anymore.) See Figure 15.
- Now the only unsolved subproblems are subproblems 3 and 5. We consider subprogram 5 by the LIFO rule.

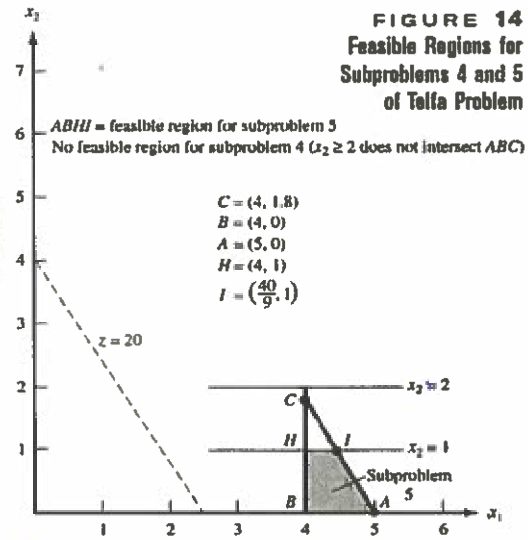


FIGURE 14
Feasible Regions for Subproblems 4 and 5 of Telfa Problem

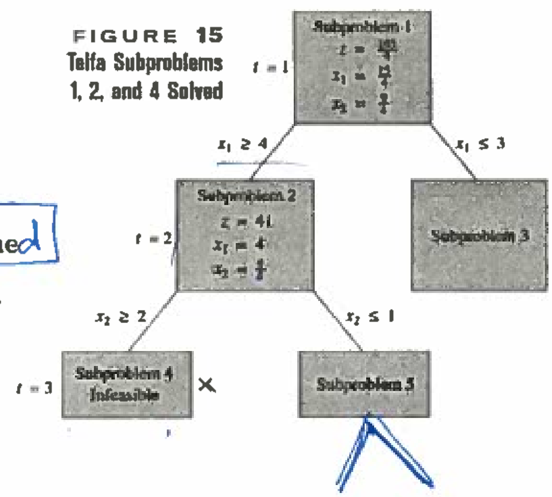


FIGURE 15
Telfa Subproblems 1, 2, and 4 Solved

- From Figure 14, we see that the optimal solution to subproblem 5 is point I in Figure 14: $z = 365/9, x_1 = 40/9, x_2 = 1$.
- So we choose to partition subproblem 5s feasible region by branching on the fractional-valued variable x_1 two new subproblems

Subproblem 6 Subproblem 5 + Constraint $x_1 \geq 5$.

Subproblem 7 Subproblem 5 + Constraint $x_1 \leq 4$.

- Subproblems 6 and 7 include all integer points that were included in the feasible region for subproblem 5.
- No point having $x_1 = 40/9$ can be in the feasible region for subproblem 6 or subproblem 7.
- The optimal solution to subproblem 5 will not recur when we solve subproblems 6 and 7.
- Our tree now looks as shown in Figure 17.
- Subproblems 3, 6, and 7 are now unsolved.
- The LIFO rule implies that we next solve subproblem 6 or subproblem 7. We solve subproblem 7.
- From Figure 16, we see that the optimal solution to subproblem 7 is point $H: z = 37, x_1 = 4, x_2 = 1$.
- Both x_1 and x_2 assume integer values, so this solution is feasible for the original IP.
- We now know that subproblem 7 yields a feasible integer solution with $z = 37$.
- We also know that subproblem 7 cannot yield a feasible integer solution having $z = 37$.
- Thus, further branching on subproblem 7 will yield no new information about the optimal solution to the IP, and subproblem has been fathomed.
- The tree to date is pictured in Figure 18.

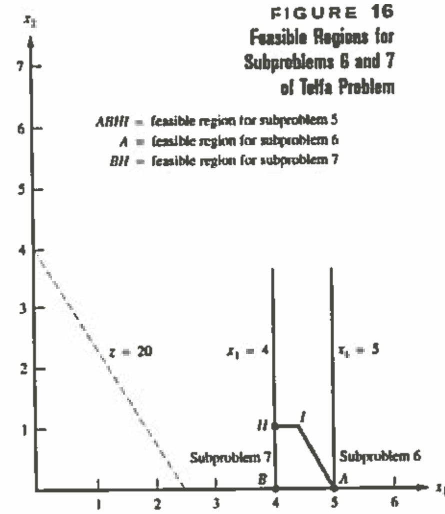


FIGURE 16
Feasible Regions for Subproblems 6 and 7 of Telfa Problem

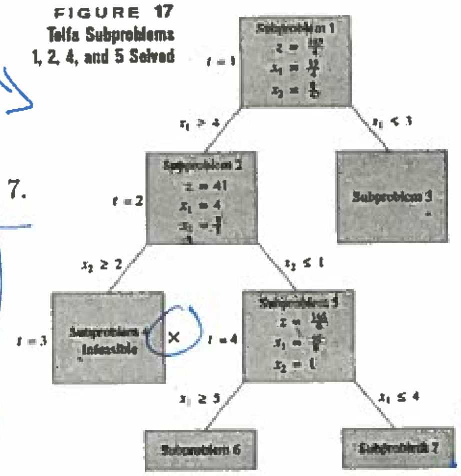


FIGURE 17
Telfa Subproblems 1, 2, 4, and 5 Solved

$z = 37$
$x_1 = 4$
$x_2 = 1$

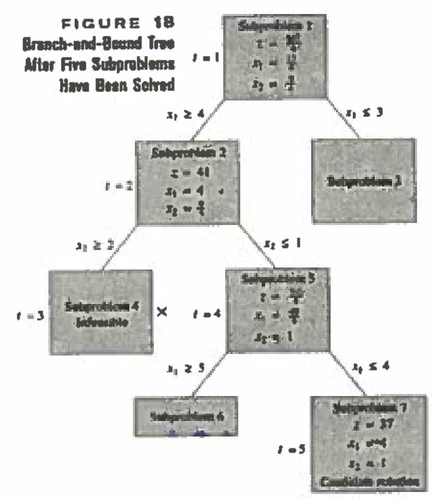


FIGURE 18
Branch-and-Bound Tree After Five Subproblems Have Been Solved

- A solution obtained by solving a subproblem in which all variables have integer values is a candidate solution.

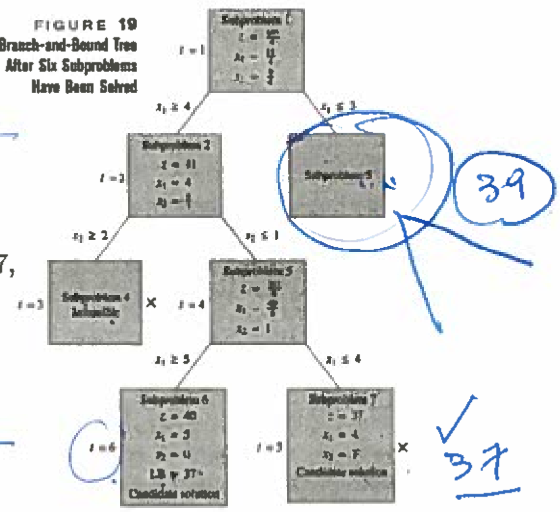


FIGURE 19 Branch-and-Bound Tree After Six Subproblems Have Been Solved

- Because the candidate solution may be optimal, we must keep a candidate solution until a better feasible solution to the IP (if any exists) is found.
- We have a feasible solution to the original IP with $z = 37$, so the optimal z-value for the IP is 37.
- Thus, the z-value for the candidate solution is a lower bound on the optimal z-value for the original IP.
- We note this by placing the notation LB is 37 in the box corresponding to the next solved subproblem (see Figure 19).
- The only remaining unsolved subproblems are 6 and 3.
- Following the LIFO rule, we next solve subproblem 6.

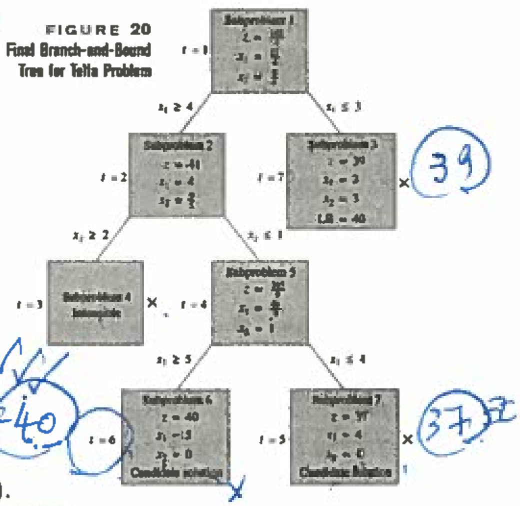


FIGURE 20 Final Branch-and-Bound Tree for Telfa Problem

- From Figure 16, we find that the optimal solution to subproblem 6 is point A: $z = 40, x_1 = 5, x_2 = 0$.
- All decision variables have integer values, so this is a candidate solution.
- Its z-value of 40 is larger than the z-value of the best previous candidate (candidate 7 with $z = 37$).
- Thus, subproblem 7 cannot yield the optimal solution of the IP (we denote this fact by placing an $\times$ by subprblem 7). We also update our LB to 40. (See Figure 20).
- Subproblem 3 is the only remaining unsolved problem.
- From Figure 12, the optimal solution to subproblem 3 is point $F : z = 39, x_1 = x_2 = 3$.
- Subproblem 3 cannot yield a z-value exceeding the current lower bound of 40, so it cannot yield the optimal solution to the original IP.
- Therefore, we place an $\times$ by it in Figure 20. From Figure 20, there are no remaining unsolved subproblems, and that only subproblem 6 can yield the optimal solution to the IP.
- Thus, the optimal solution to the IP is for Telfa to manufacture 5 tables and 0 chairs.
- This solution will contribute \$40 to profits.

- In using the branch-and-bound method to solve the Telfa problem, we have implicitlyenumerated all points in the IPs feasible region.
- Eventually, all such points (except for the optimal solution) are eliminated from consideration, and the branch-and-bound procedure is complete.
- To show that the branch-and-bound procedure actually does consider all points in the IPs feasible region, we examine several possible solutions to the Telfa problem and show how the procedure found these points to be nonoptimal.
- For example, how do we know that $x_1 = 2, x2 = 3$ is not optimal?
- This point is in the feasible region for subproblem 3, and we know that all points in the feasible region for subproblem 3 have $z = 39$.
- Thus, our analysis of subproblem 3 shows that $x_1 = 2, x_2 = 3$ cannot beat $z = 40$ and cannot be optimal.
- As another example, why isnt $x_1 = 4, x_2 = 2$ optimal?
- Following the branches of the tree, we find that $x_1 = 4, x_2 = 2$ is associated with subproblem 4.
- Because no point associated with subproblem 4 is feasible, $x_1 = 4, x_2 = 2$ must fail to satisfy the constraints for the original IP and thus cannot be optimal for the Telfa problem.
- In a similar fashion, the branch-and-bound analysis has eliminated all points x_1, x_2 (except for the optimal solution) from consideration.
- For the simple Telfa problem, the use of the branch-and-bound method may seem like using a cannon to kill a fly.
- But for an IP in which the feasible region contains a large number of integer points, the procedure can be very efficient for eliminating nonoptimal points from consideration.
- For example, suppose we are applying the branch-and-bound method and our current LB is 42.
- Suppose we solve a subproblem that contains 1 million feasible points for the IP.
- If the optimal solution to this subproblem has $z = 42$, then we have eliminated 1 million nonoptimal points by solving a single LP!
- The key aspects of the branch-and-bound method for solving pure IPs (mixed IPs are considered in the next section) may be summarized as follows:

Step 1 If it is unnecessary to branch on a subproblem, then it is fathomed. The following three situations result in a subproblem being fathomed:

(1) The subproblem is infeasible;

(2) the subproblem yields an optimal solution in which all variables have integer values; and

(3) the optimal z-value for the subproblem does not exceed (in a max problem) the current LB.

Step 2 A subproblem may be eliminated from consideration in the following situations:

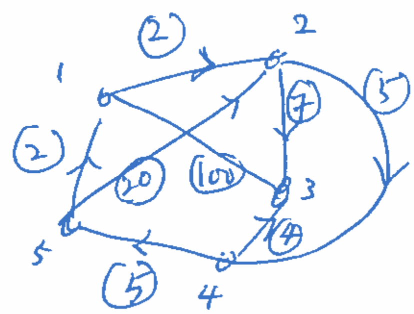
(1) The subproblem is infeasible (in the Telfa problem, subproblem 4 was eliminated for this reason);

(2) the LB (representing the z-value of the best candidate to date) is at least as large as the z-value for the subproblem (in the Telfa problem, subproblems 3 and 7 were eliminated for this reason).

Formulation of Problem could be ~~different~~ difficult :

Problem: Given a "strongly connected network" with costs assigned to the arcs.

Example:



Question ⓪ Find the minimum (cost) subnetwork of the graph that is still strongly connected.

ⓐ Find a ~~graph~~ (MCNF) network formulation for the problem.

ⓑ Would the greedy algorithm work?
That is, removing "expensive" arc and maintaining strongly connectedness.

(10 points for mid-term grade.)

Example 6 Either Or Constraints Dorian Auto is considering manufacturing three types of autos: compact, midsize, and large.

The resources required for, and the profits yielded by, each type of car are shown in Table 8.

Currently, 6,000 tons of steel and 60,000 hours of labor are available.

For production of a type of car to be economically feasible, at least 1,000 cars of that type must be produced.

Formulate an IP to maximize Dorians profit.

Let x_1, x_2, x_3 be the number of compact, midsize, large cars produced.

We know that if any cars of a given type are produced, then at least 1,000 cars of that type must be produced. Thus, for $i = 1, 2, 3$, we must have $x_i \le 0$ or $x_i \ge 1{,}000$. Steel and labor are limited, so Dorian must satisfy the following five constraints:

Constraint 1 $x_1 \le 0$ or $x_1 \ge 1{,}000$.

Constraint 2 $x_2 \le 0$ or $x_2 \ge 1{,}000$.

Constraint 3 $x_3 \le 0$ or $x_3 \ge 1{,}000$.

Constraint 4 The cars produced can use at most 6,000 tons of steel.

Constraint 5 The cars produced can use at most 60,000 hours of labor.

We may replace Constraint 1 by the following:

$$\boxed{x_1 \le M_1 y_1}$$
$$1{,}000 - x_1 \le M_1(1 - y_1)$$
$$y_1 = 0 \text{ or } 1$$

TABLE 8

Resources and Profits for Three Types of Cars

	Car Type		
Resource	**Compact**	**Midsize**	**Large**
Steel required	1.5 tons	3 tons	5 tons
Labor required	30 hours	25 hours	40 hours
Profit yielded ($)	2,000	3,000	4,000

To ensure that both x_1 and $1{,}000 - x_1$ will never exceed M_1, it suffices to choose M_1 large enough so that M_1 exceeds 1,000 and x_1 is always less than M_1. Building $\frac{60{,}000}{30} = 2{,}000$ compacts would use all available labor (and still leave some steel), so at most 2,000 compacts can be built. Thus, we may choose $M_1 = 2{,}000$.

We can apply similar argument to the second and third constraints and get the following.

$$
\begin{aligned}
\max z &= 2x_1 + 3x_2 + 4x_3 \\
\text{s.t.} \quad x_1 &\le 2{,}000y_1 \\
1{,}000 - x_1 &\le 2{,}000(1 - y_1) \\
x_2 &\le 2{,}000y_2 \\
1{,}000 - x_2 &\le 2{,}000(1 - y_2) \\
x_3 &\le 1{,}200y_3 \\
1{,}000 - x_3 &\le 1{,}200(1 - y_3) \\
1.5x_1 + 3x_2 + 5x_3 &\le 6{,}000 \quad \text{(Steel constraint)} \\
30x_1 + 25x_2 + 40x_3 &\le 60{,}000 \quad \text{(Labor constraint)} \\
x_1, x_2, x_3 &\ge 0;\ x_1, x_2, x_3 \text{ integer} \\
y_1, y_2, y_3 &= 0 \text{ or } 1
\end{aligned}
$$

The optimal solution to the IP is $z = 6{,}000$, $x_2 = 2{,}000$, $y_2 = 1$, $y_1 = y_3 = x_1 = x_3 = 0$. Thus, Dorian should produce 2,000 midsize cars. If Dorian had not been required to manufacture at least 1,000 cars of each type, then the optimal solution would have been to produce 570 compacts and 1,715 midsize cars.

If–Then Constraints

In many applications, the following situation occurs: ~~We want to ensure that~~ if a constraint $f(x_1, x_2, \ldots, x_n) > 0$ is satisfied, then the constraint $g(x_1, x_2, \ldots, x_n) \geq 0$ must be satisfied, while if $f(x_1, x_2, \ldots, x_n) > 0$ is not satisfied, then $g(x_1, x_2, \ldots, x_n) \geq 0$ may or may not be satisfied. In short, we want to ensure that $f(x_1, x_2, \ldots, x_n) > 0$ implies $g(x_1, x_2, \ldots, x_n) \geq 0$.

To ensure this, we include the following constraints in the formulation:

$$-g(x_1, x_2, \ldots, x_n) \leq My \tag{28}$$

$$f(x_1, x_2, \ldots, x_n) \leq M(1 - y) \tag{29}$$

$$y = 0 \text{ or } 1$$

As usual, M is a large positive number. (M must be chosen large enough so that $f \leq M$ and $-g \leq M$ hold for all values of $x_1, x_2, \ldots, x_n$ that satisfy the other constraints in the problem.)

Example

To illustrate the use of this idea, suppose we add the following constraint to the Nickles lockbox problem: If customers in region 1 send their payments to city 1, then no other customers may send their payments to city 1. Mathematically, this restriction may be expressed by

$$\text{If } x_{11} = 1, \qquad \text{then} \qquad x_{21} = x_{31} = x_{41} = 0 \tag{30}$$

Because all x_{ij} must equal 0 or 1, (30) may be written as

$$\text{If } x_{11} > 0, \qquad \text{then} \qquad x_{21} + x_{31} + x_{41} \leq 0, \qquad \text{or} \qquad -x_{21} - x_{31} - x_{41} \geq 0 \tag{30'}$$

If we define $f = x_{11}$ and $g = -x_{21} - x_{31} - x_{41}$, we can use (28) and (29) to express (30′) [and therefore (30)] by the following two constraints:

$$x_{21} + x_{31} + x_{41} \leq My$$

$$x_{11} \leq M(1 - y)$$

$$y = 0 \text{ or } 1$$

Because $-g$ and f can never exceed 3, we can choose $M = 3$ and add the following constraints to the original lockbox formulation:

$$x_{21} + x_{31} + x_{41} \leq 3y$$

$$x_{11} \leq 3(1 - y)$$

$$y = 0 \text{ or } 1$$