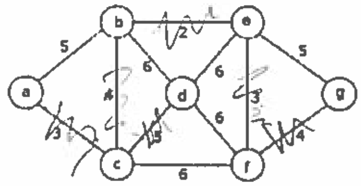
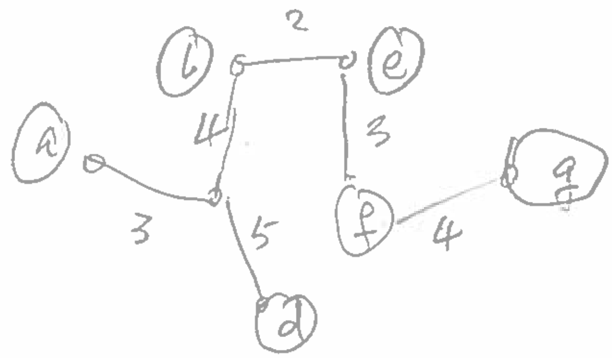
Sample Solution.

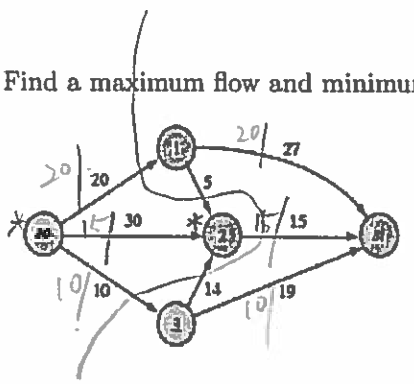
Math 323 QUIZ 5 Name: ____________________

1. Find a mimimum spanning tee of the following network.





2. Find a maximum flow and minimum cut in the following network.



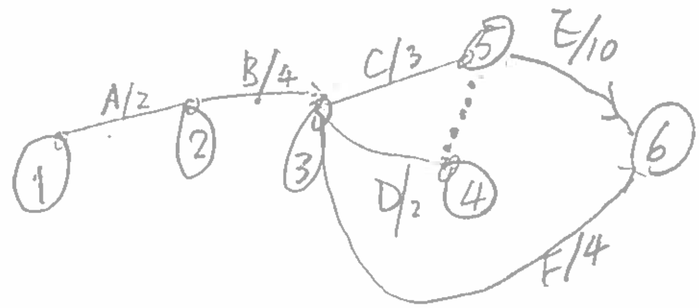
Flow 45

$S = \{50, 2\}$

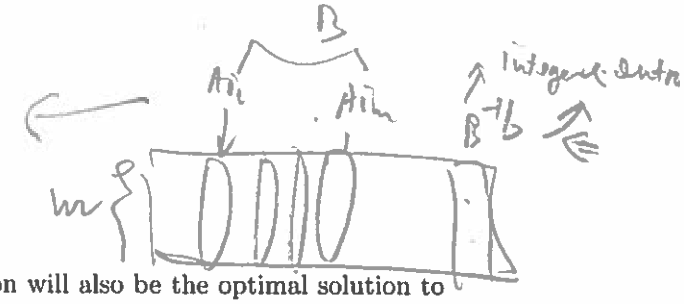
$K(S, \bar{S}) = 20 + 15 + 10 = 45$

3. Horizon Cable is about to expand its cable TV offerings in Smalltown. The activities in the following table must be completed before the service expansion is completed.
Draw the project network of the activities.

Activity	Description	Immediate Predecessors	Duration (Weeks)
A	Choose stations	—	2
B	Get town council to approve expansion	A	4
C	Order converters needed to expand service	B	3
D	Install new dish to receive new stations	B	2
E	Install converters	C, D	10
F	Change billing system	B	4



$B^{-1} = \frac{adj(B)}{det(B)}$



Remarks

1. For some IPs, the optimal solution to the LP relaxation will also be the optimal solution to the IP. Suppose the constraints of the IP are written as $Ax = b$. If the determinant of every square submatrix of A is $1, -1$, or 0, we say that the matrix A is **unimodular**.

   If A is unimodular and each element of b is an integer, then the optimal solution to the LP relaxation will assign all variables integer values and will therefore be the optimal solution to the IP.

   For example, the constraint matrix of any MCNFP is unimodular. Thus, as was discussed in Chapter 8, any MCNFP in which each nodes net outflow and each arcs capacity are integers will have an integer-valued solution.

2. As a general rule, the more an IP looks like an MCNFP, the easier the problem is to solve by branch-and-bound methods. Thus, in formulating an IP, it is good to choose a formulation in which as many variables as possible have coefficients of $1, -1$, and 0.

   To illustrate this idea, recall that the formulation of the Nickles (lockbox) problem given in Section 9.2 contained 16 constraints of the following form:

   Formulation 1 $\quad x_{ij} \leq y_j \quad (i = 1, 2, 3, 4; j = 1, 2, 3, 4)$.

   As we have already seen in Section 9.2, if the 16 constraints in above are replaced by the following 4 constraints, then an equivalent formulation results:

   Formulation 2

   $$x_{11} + x_{21} + x_{31} + x_{41} \leq 4y_1 \qquad x_{12} + x_{22} + x_{32} + x_{42} \leq 4y_2$$
   $$x_{13} + x_{23} + x_{33} + x_{43} \leq 4y_3 \qquad x_{14} + x_{24} + x_{34} + x_{44} \leq 4y_2$$

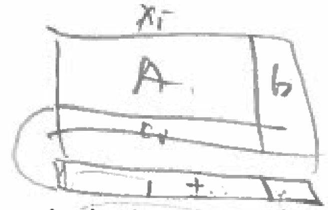
   Because formulation 2 has $16 - 4 = 12$ fewer constraints than formulation 1, one might think that formulation 2 would require less computer time to find the optimal solution. This turns out to be untrue.

   Reason. The feasible region of the LP relaxation of formulation 2 contains many more non-integer points than the feasible region of formulation 1.

   For example, the point $y_1 = y_2 = y_3 = y_4 = 1/4$, $x_{11} = x_{22} = x_{33} = x_{44} = 1$ and $x_{ij} = 0$ for other i, j is in the feasible region for the LP relaxation of formulation 2, but not for formulation 1.

3. When solving an IP in the real world, we are usually happy with a near-optimal solution. For example, if we can find a feasible solution with z-value closed to the LP relaxation, say, the ratio of the two optimal values is larger than .9, then we might want to save the effort, computer time, real time for making decision, and just use this feasible solution.

   For this reason, the branch-and-bound procedure is often terminated when a candidate solution is found with a z-value close to the z-value of the LP relaxation.



$X_i \geq k$
$\leq$

$Ax = b$

4. Subproblems for branch-and-bound problems are often solved using some variant of the dual simplex algorithm because we always add a constraint to generate new subprogram.

5. Sometimes, one can show that if the constraints $x_k = i$ and $x_k = i+1$ are added, then the optimal solution to the first subproblem will have $x_k = i$ and the optimal solution to the second subproblem will have $x_k = i+1$. This observation is very helpful, especially, when we graphically solve subproblems.

9.4 The Branch-and-Bound Method for Mixed Integer Programming Problems

- In a mixed IP, some variables are required to be integers and others are allowed to be either integers or nonintegers.
- To solve a mixed IP by the branch-and-bound method, modify the method described in Section 9.3 by branching only on variables that are required to be integers.

Example $\max Z = 2x1 + x2$

s.t. $5x_1 + 2x_2 = 8$

$x1 + x2 \leq 3$

$x_1, x_2 \geq 0$; x_1 integer

- We begin by solving the LP relaxation of the IP.
- The optimal solution of the LP relaxation is $Z = 11/3, x_1 = 2/3, x_2 = 7/3$.
- Because x_2 is allowed to be fractional, we do not branch on x_2.
- So, we branch at x_1 and solve subproblem 2.
- The optimal solution to subproblem 2 is the candidate solution $z = 3, x_1 = 0, x_2 = 3$.

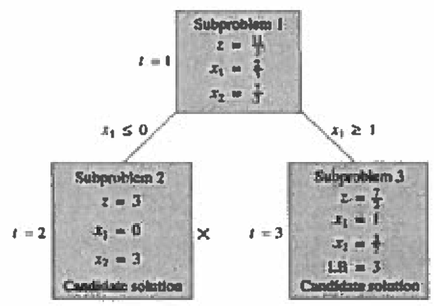


FIGURE 21
Branch-and-Bound Tree for Mixed IP

- We solve subproblem 3 and obtain the candidate solution $z = 7/2, x_1 = 1, x_2 = 3/2$.
- The z-value from the subproblem 3 candidate exceeds the z-value for the subproblem 2 candidate, so subproblem 2 can be eliminated from consideration, and the subproblem 3 candidate $(z = 7/2, x_1 = 1, x_2 = 3/2)$ to the mixed IP.

Solving Knapsack Problems by the Branch-and-Bound Method

- A knapsack problem is an IP with a single constraint, and each variable must equal 0 or 1.
- So, a knapsack problem may be written as

$$\max z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$
$$\text{s.t.} \quad a_1x_1 + a_2x_2 + a_nx_n \leq b$$
$$x_i \in \{0,1\} \quad (i = 1, 2, ..., n).$$

- Recall that c_i is the benefit obtained if item i is chosen, b is the amount of an available resource, and a_i is the amount of the available resource used by item i.
- When knapsack problems are solved by the branch-and-bound method, two aspects of the method greatly simplify.

(1) Because x_i must equal 0 or 1, branching on x_i will yield an $x_i = 0$ and an $x_i = 1$ branch.

(2) Also, the LP relaxation (and other subproblems) may be solved by inspection.

Reason. a_i/c_i may be interpreted as the benefit item i earns for each unit of the resource used by item i. Thus, the best (worst) items have the largest (smallest) values of a_i/c_i. So, one can choose items with large c_i/a_i values.

Example Consider the problem:

$$\max z = 40x_1 + 80x_2 + 10x_3 + 10x_4 + 4x_5 + 20x_6 + 60x_7$$
$$\text{s.t.} \quad 40x_1 + 50x_2 + 30x_3 + 10x_4 + 10x_5 + 40x_6 + 30x_7 \leq 100$$
$$x_i \in \{0,1\} \quad (i = 1, 2, ..., 7).$$

We begin by computing the a_i/c_i ratios and ordering the variables from best to worst.

i	1	2	3	4	5	6	7
c_i/a_i	1	8/5	1/3	1	4/10	1/2	2

Then choose item 7, and $100 - 30 = 70$ units of the resource remain.
Then choose 2, and $70 - 50 = 20$ units of the resource remain.
Then choose item 4 or item 1; we choose item 4, and $20 - 10 = 10$ units of the resource remain.
The best remaining item is item 1. We fill the knapsack with as much of item 1 as we can.
Because only 10 units of the resource remain, we set $x1 = 1/4$ (for the LP relaxation problem).
Thus an optimal solution to the LP relaxation is $z = 80 + 60 + 10 + (1/4)40 = 160$ with $x_7 = x_2 = x_4 = 1$ and $x_1 = 1/4$.

Example 1 Stockco capital budgeting problem

$$\max z = 16x_1 + 22x_2 + 12x_3 + 8x_4$$

$$\text{s.t.} \quad 5x_1 + 7x_2 + 4x_3 + 3x_4 \le 14$$

$$x_1, \ldots, x_4 \in \{0, 1\}.$$

The branch-and-bound tree for this problem is shown in the following Figure.

The optimal solution is $z = 42$, with $x_1 = 0, x2 = x3 = x4 = 1$.