Minimax Algorithm

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Definition

- Minimax is a algorithm that is used in decision making and game theory to find the optimal move for a player.
- It is widely used in two player turn-based games such as Tic-Tac-Toe, rock-paper-scissors Chess, etc.
- assumption: both player play optimally.

Roles

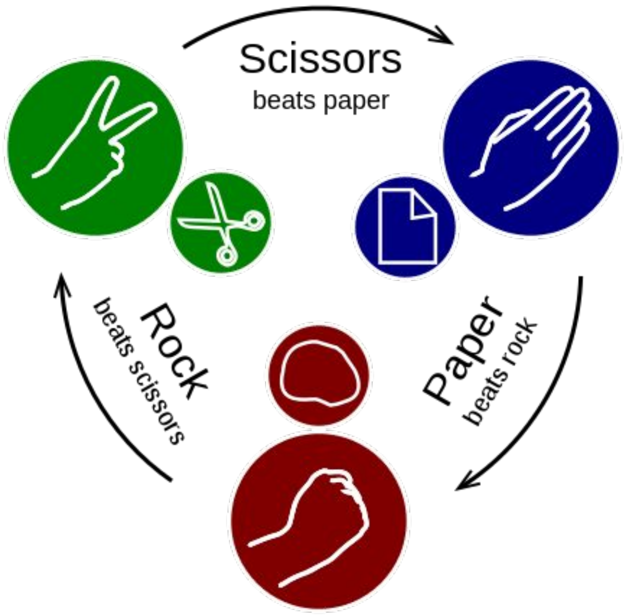
- The **maximizer** tries to get the highest score possible, i.e., maximize his/her gains.
- the **minimizer** tries to do the opposite, i.e., minimize his/her losses.

Zero-sum Game

- For one players to win, another must be lose
- Total utility is divided among player
- So the payoff is always balanced
- *prisoners' dilemma is not a zero-sum game

Rock-Paper-Scissors

- Rock beats Scissors
- Scissors beats Paper
- Paper beats Rock



Payoff Matrix

		Player 2		
		Rock	Paper	Scissors
Player 1	Rock	0	-1	1
	Paper	1	0	-1
	Scissors	-1	1	0

Best strategy?

- https://www.youtube.com/watch?v=AnRYS02tvRA

Find best strategy using Minimax Theorem

- Minimax theorem (von Neumann, 1928): In any finite, two-player, zero-sum game, in any Nash equilibrium each player receives a payoff that is equal to both his maxmin value and his minmax value.

- Maximin strategy is a strategy that maximizes one's worst-case payoff, i.e., the minimum amount of payoff guaranteed by a maximin strategy

- Minimax strategy is a strategy that minimizes the opponent's best-case payoff.

Maximin

- Let Xi = the probability that player A chooses action i, for i ∈ {Rock,Paper, Scissors}.
- Then Player A’s maximin strategy can be found with the following optimization model:
-

$$\text{maximize } \min\{x2 - x3,\ x3 - x1,\ x1 - x2\}$$
$$\text{S.t.} \quad x1 + x2 + x3 = 1$$
$$x1 \geq 0,\ x2 \geq 0,\ x3 \geq 0$$

This model is equivalent to the following linear program:

$$
\begin{aligned}
& \text{maximize } z \\
\text{st:} \quad & z \le x2 - x3 \\
& z \le x3 - x1 \\
& z \le x1 - x2 \\
& x1 + x2 + x3 = 1 \\
& x1 \ge 0,\ x2 \ge 0,\ x3 \ge 0
\end{aligned}
$$

Rewritten LP for Maximin:

By putting all the decision variables are on the left hand side of the constraints, and all the constants are on the right hand side:

$$
\begin{aligned}
&\text{maximize } z \\
\text{s.t. } \quad & z - x2 + x3 \leq 0 \\
& z + x1 - x3 \leq 0 \\
& z - x1 + x2 \leq 0 \\
& x1 + x2 + x3 = 1 \\
& x1 \geq 0, x2 \geq 0, x3 \geq 0
\end{aligned}
$$

Minimax

- Let vi = the probability the player B chooses action i, for i ∈ {Rock,Paper, Scissors}.
- Then Player B's minimax strategy can be found with the following optimization model:

$$\text{minimize } \max\{-v2 + v3,\ v1 - v3,\ -v1 + v2\}$$
$$\text{subject to } v1 + v2 + v3 = 1$$
$$v1 \geq 0,\ v2 \geq 0,\ v3 \geq 0$$

This model is equivalent to the following linear program:

$$\begin{aligned} &\text{minimize } w \\ \text{St.}\quad & w \geq -v2 + v3 \\ & w \geq v1 - v3 \\ & w \geq -v1 + v2 \\ & v1 + v2 + v3 = 1 \\ & v1 \geq 0,\ v2 \geq 0,\ v3 \geq 0 \end{aligned}$$

Dualization of Maxmin

Primal:

maximize z

$$
\begin{aligned}
\text{s.t.}\quad & z - x2 + x3 \le 0 \ (v1) \\
& z + x1 - x3 \le 0 \ (v2) \\
& z - x1 + x2 \le 0 \ (v3) \\
& x1 + x2 + x3 = 1 \ (w) \\
& x1 \ge 0,\ x2 \ge 0,\ x3 \ge 0
\end{aligned}
$$

→

Dual:

minimize w

$$
\begin{aligned}
\text{S.t.}\quad & v1 + v2 + v3 = 1 \\
& v2 - v3 + w \ge 0 \\
& v3 - v1 + w \ge 0 \\
& v1 - v2 + w \ge 0 \\
& v1 \ge 0,\ v2 \ge 0,\ v3 \ge 0,\ w \text{ free}
\end{aligned}
$$

Dual of Player A

minimize w

S.t. $v1 + v2 + v3 = 1$

$v2 - v3 + w \geq 0$

$v3 - v1 + w \geq 0$

$v1 - v2 + w \geq 0$

$v1 \geq 0, v2 \geq 0, v3 \geq 0$, w free

=

Primal of Player B

minimize w

S.t. $w \geq -v2 + v3$

$w \geq v1 - v3$

$w \geq -v1 + v2$

$v1 + v2 + v3 = 1$

$v1 \geq 0, v2 \geq 0, v3 \geq 0$

Strong Duality theorem

- If either Primal or Dual has a finite optimal value, then so does the other, and optimal solutions to both P and D exist and are the same.

- Therefore, solution of Maximin = solution of Minimax

AMPL Code and Solution

```
var x1 >= 0;
var x2 >= 0;
var x3 >= 0;
var w;

minimize objective: w;
subject to cons1: x1+x2+x3=1;
subject to cons2: x2-x3+w>=0;
subject to cons3: x3-x1+w>=0;
subject to cons4: x1-x2+w>=0;
```

```
CPLEX 12.8.0.0: optimal solution; objective 0
3 dual simplex iterations (2 in phase I)
ampl: display x1;
x1 = 0.333333

ampl: display x2;
x2 = 0.333333

ampl: display x3;
x3 = 0.333333
```

Solution for playerA: x1= ⅓, x2=⅓, x3=⅓
Solution for playerB: v1= ⅓, v2=⅓, v3=⅓

Conclusion

By using Minimax Theorem, we proved that “play each alternative with probability 1/3”strategy is optimal for both player A and player B.

However, there are problems

- Mixed strategies may be implausible. It is counterintuitive that players choose their strategies based only on their opponents' payoffs.

- Genuine random selection of actions is almost impossible and unsuccessful randomization leads to predictability.

- The equilibrium prescribed by the Minimax Theorem is highly unstable.

Importance

- The Minimax Theorem gave birth to the theory of duality (of LP)
- Guide decision making process
- Has an strong impact on the social sciences including economics, political science and psychology.
- The Minimax Theorem provides us with the tools to analyze payoffs of various situations and really enable people to make better decision.

Reference

1. https://www.youtube.com/watch?v=AnRYS02tvRA
2. https://www.geeksforgeeks.org/minimax-algorithm-in-game-theory-set-1-introduction/
3. https://www.cs.ubc.ca/~kevinlb/teaching/cs532l%20-%202007-8/lectures/lect4.pdf
4. https://www.usna.edu/Users/math/dphillip/sa305.s15/phillips/solutions/rock-paper-scissors.sol.pdf
5. http://www.math.mcgill.ca/vetta/CS360.dir/SimplexLec3.pdf

Q&A

Thank you!