Projections: The Mathematics of Mapping

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Maps are one of the most common mediums humans use to explore and explain the world. One of the biggest problems with maps, however, is that they can lie. Distortion is a necessary part of representing a three-dimensional surface on a two-dimensional plane and we must be aware of what distortions occur in order to make sure we are using a map as it is intended. In this paper, I will prove why distortion is inherent to the mapping process, describe some of the most common projection types, and take a look at some of the mathematics involved in map projections. Last, I will reflect on classmate feedback and discussion to introduce areas of improvement for future presentations.

Definitions

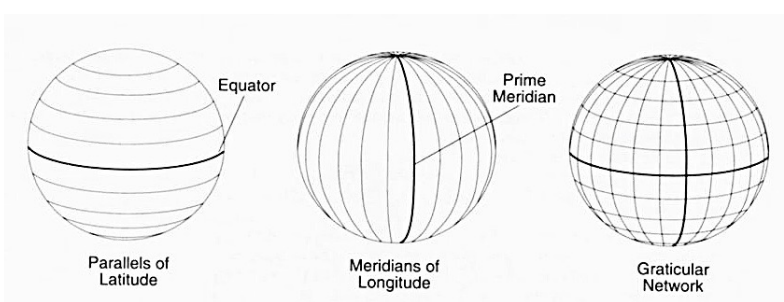

Figure 1: Definition Diagram

Longitude: Meridians of longitude extend from the north to the south poles. All locations on the same line of longitude experience noon at the same time. The Prime Meridian is defined to be at 0 degrees longitude and distances east and west on the globe are measured in degrees from the Prime Meridian. At 180 degrees from the Prime Meridian is the International Date Line, where places are a full day apart (Feeman).

Latitude: Parallels of latitude are horizontal rings around the globe. The Equator, the most famous parallel of latitude, goes around the middle of the globe - it is defined to be at 0 degrees latitude. Distances north and south on the globe are measured from the Equator with the degree of latitude equal to the difference in the angle of the sun at noon at that location from the place on the Equator located on the same line of longitude (Feeman).

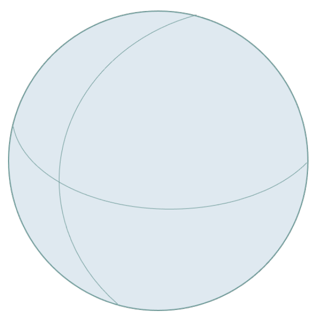

Figure 2: Great Circle Diagram

Great Circle: A great circle is any circle created on the surface of a sphere by a plane that goes through the center of the sphere. Thus, each great circle has a radius equal to that of the sphere. As the curvature of a circle is equal to the reciprocal of the radius, these circles are the "straightest" circles that can be drawn on the surface of a sphere. Examples of great circles on the globe are meridians and the Equator. It is important to note that parallels, other than the Equator, are not great circles as their center points are either above or below the center of the globe (Feeman).

Projection: A projection is "any systematic representation of the earth's surface onto another surface." As maps represent the three dimensional surface of the Earth on a two dimensional plane, projections are a necessary part of cartography.

The Ideal Projection

When creating a new map, one of the most important things to think about is what factors need to be preserved from the globe to the map. The four most important factors are direction, distance, angle, and area - so in an ideal map projection each of these would be correctly mapped from the sphere to the plane. We can think of this mapping as a function from S^2 to R^2. In a perfect world the function would be an isometry preserving the four "relevant geometric features," but I will prove that no such function exists (McCleary).

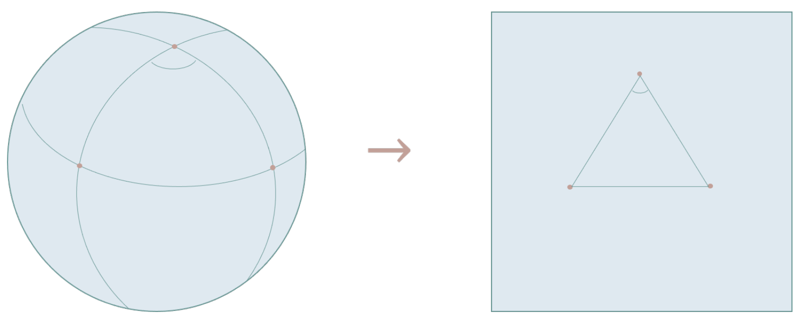

Figure 3: Mapping the Sphere to the Plane

Proof. Consider a triangle created by the intersection of three great circles of a sphere. By definition of a great circle, these represent the "straightest" curves of a sphere. It follows that the shortest distance from one point of the triangle to another is along the great circle. Therefore, in order for distance to preserved, the great circles must be mapped to straight lines on the plane as shown in Fig.3 above. Now consider the angles of the triangle on the surface of the sphere. The sum of the angles of a triangle on the sphere can range from $(\pi, 3\pi)$. However, the sum of the angles of a triangle on a plane is exactly π. Thus when distance is preserved, angle is not (Feeman). $\square$

This simple proof describes one of the biggest problems that cartographers face - map distortion. In order to preserve one attribute of the globe, another must be distorted. One tool we can use to denote distortion in a projection is Tissot's Indicatrix. This indicatrix compares scale factors along principal directions of a map, such as parallels and meridians.

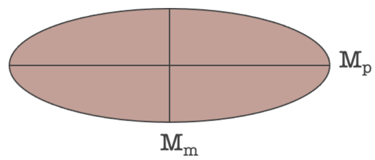

Figure 4: Tissot's Indicatrix

When the scale factors are the same along principal directions this ellipse will appear as a circle. Otherwise it's size or shape will be distorted to show changes in the four projection factors (Feeman).

Azimuthal Projections

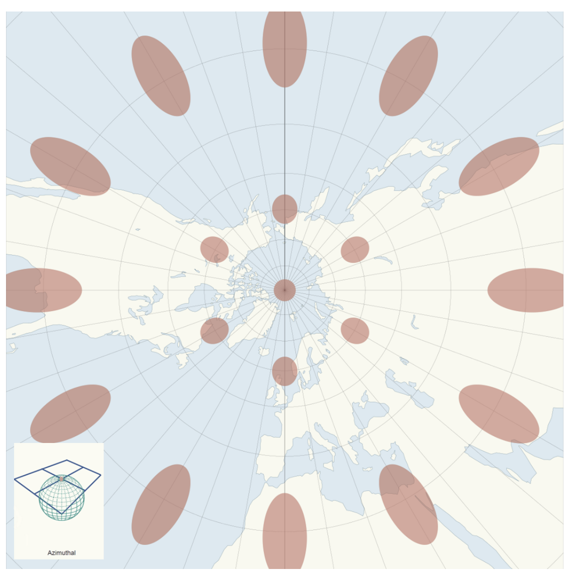

Figure 5: Gnomonic Projection

One of the most basic projection types is the Azimuthal Projection, also known as the planar projection. This projection can be achieved by placing a plane tangent to the globe. A light source is chosen and the shadows of the meridians and parallels are traced onto the plane. In particular, the map shown is a gnomonic projection where the plane is tangent to the north pole and the light source is located at the south pole - this transfers the meridians and parallels of only the northern hemisphere. The point where the plane is tangent to the surface of the globe is the place where there is no distortion. The amount of distortion in the map increases as we move away from this point. Note that in this projection all of the great circles are projected to straight lines. Thus this projection is distance preserving and all straight-line distances are the shortest distances between points.

Conical Projection

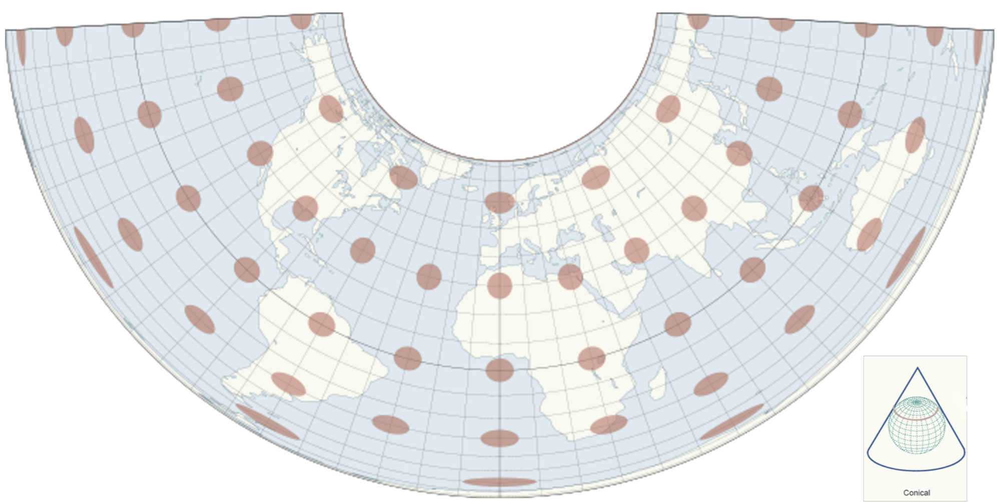

Figure 6: Alber's Equal Area Conic Projection

Another common type of projection is the conical projection. As the name suggests, this projection is achieved by wrapping a cone around the globe - the parallels are then pulled horizontally onto the surface of the cone. The map featured is Alber's Equal Area Conic - an area preserving map. Looking at the Tissot's Indicatrix, each ellipse while not the same shape, does encompass the same amount of area. As this map preserves areas, it would be a good map to use when viewing the world in order to compare sizes of the various land masses (Borneman).

Cylindrical Projection

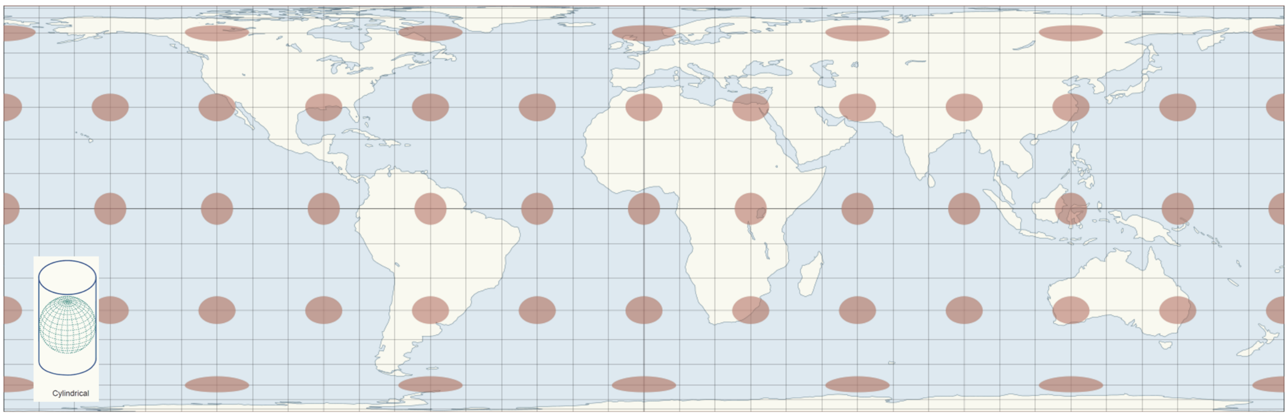

Figure 7: Lambert's Cylindrical Equal Area Projection

Cylindrical projections are often what we think about when we picture a map. These projections are rectangular depictions of the Earth, produced by wrapping a cylinder around the globe and pulling parallels and meridians horizontally onto the surface of the cylinder. The area with the least amount of distortion is the line of the map that corresponds with the great circle of the globe where the cylinder is tangent with the surface. The map shown is Lambert's Cylindrical Equal Area Projection and the Equator represents the place where this map would be tangent with the globe. As you move north or south on the map, the vertical distance between parallels of latitude shrink to make up for the extended length of the parallel - this is represented in the stretched nature of Tissot's Indicatrix. Mathematically, the projection's preservation of area can be seen by comparing the area of a strip of the map formed between two parallels of latitude with the same strip on the globe.

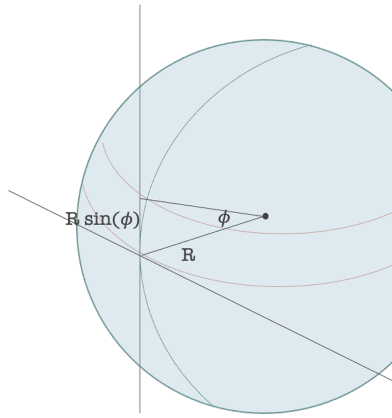

Figure 8: Equal Area Diagram

Proof. Consider the strip of the map located between the Equator and the parallel of latitude ϕ degrees north of the Equator as shown above. Note that on the map, all parallels have a length equal to $2\pi R$ where R is the radius of the globe by nature of their construction. Further, in polar coordinates, the height of the strip is equal to $Rsin(\phi)$ as the parallel ϕ is pulled horizontally onto the cylinder. Thus the area of the strip on the map is equal to $2\pi R * Rsin(\phi) = 2R^2 sin(\phi)$. On the globe, the area of this strip is equal to the surface area of a strip of a sphere or $2\pi Rh$ where h is the height of the strip. The height of the strip on the globe is also $Rsin(\phi)$ as shown in the diagram above. Therefore the area of a strip between the Equator and a parallel of latitude ϕ degrees north on the globe is equal to that of the same strip on the equal area map (Feeman). □

Pseudocylindrical Projection

The last projection I will discuss is the Pseudocylindrical Projection, a mathematically constructed projection unlike the previous projection types. It was formulated to combat the false impressions that cylindrical maps can give when it comes to the shape of the world. On the globe, the longest parallel of latitude is the Equator and parallels north and south of the Equator gradually shrink to a point at each pole.

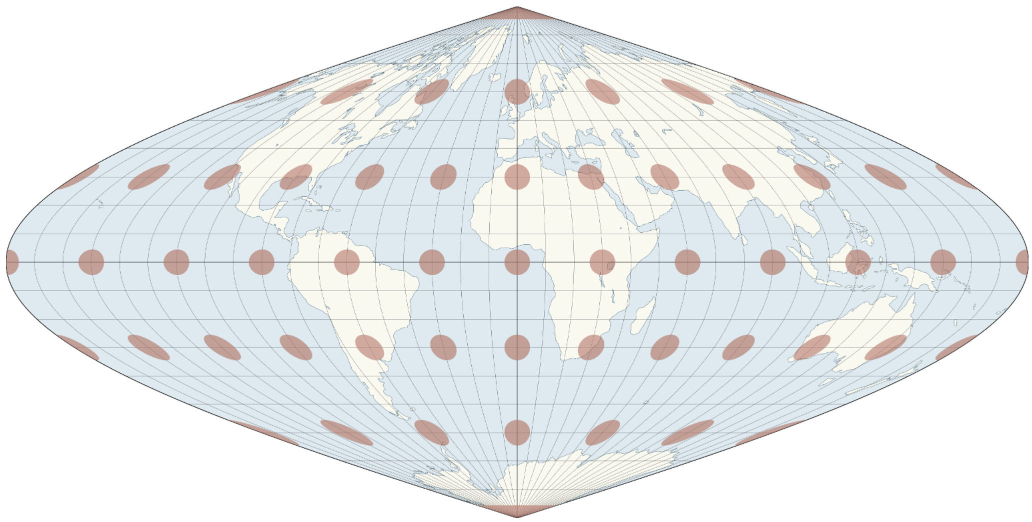

Figure 9: Sinusoidal Projection

In the Sinusoidal Projection above, this idea is transferred onto the map through the use of trigonometry to construct parallels that shrink with increased angular displacement from the Equator. Specifically, the length of a parallel ϕ degrees north of the Equator is $2\pi Rcos(\phi)$ units long. This means that at the Equator where angular displacement is 0, the length is $2\pi Rcos(0) = 2\pi R$ and at the north pole where displacement is $\frac{\pi}{2}$, the parallel is a point as its length is $2\pi cos(\frac{\pi}{2}) = 0$ (Feeman).

Reflection

I received a lot of positive feedback from my classmates about my topic choice and presentation. I think that maps are so familiar to us that we don't always stop and think about the process used to create them or the fact that maps can have a specific purpose. It was not until I took a GIS class at William and Mary that I really stopped to consider what projection I was using and how my projection type was important to my understanding of the map. Looking back at my presentation, I wish I had stressed the usage of particular maps. I think some people's take away from my presentation was that maps are bad because of their distortion, which was not my goal. I wanted the audience to know that maps distorted the world and to think about how these distortions may impact their viewpoint, but I also wanted to teach them how projections can be properly used to share information. If you understand the purpose of a map and why a projection is chosen, maps can be useful for everything from data visualization to navigation. It is only when a map is taken out of it's context that it becomes harmful. This can be seen with the Mercator map which was created for navigation, but has been adopted as the standard map used for teaching students about the world. This map was not intended as an accurate depiction continent of size and thus when used for teaching it can further a false Eurocentric viewpoint.

Works Cited

Images:

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3. Definition Image: John P. Snyder (1987) Map Projections: A Working Manual http://www.geography.ryerson.ca/michalak/geo814/tutorial_1/long-lat.html

Information

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