

Pascal's Wager: Where Decision Theory Breaks Down at Infinity

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Abstract

This paper seeks to summarize the problem associated with the underlying decision-theoretic framework of Pascal's Wager that arises with the use of infinite utility in contemporary literature. It surveys major issues that range from the violation of the axiomatic Von-Neumann Morgenstern utility theory to mixed strategies and the Many Gods Objection. The paper also offers a discussion of Alan Hájek's treatment of the problem with one of his reformulations using finite utility. We will see that Hájek's proposed reformulation of the wager failed to come to fruition in what later literature on the topic termed Hájek's dilemma. This paper focuses exclusively on conventional decision theory that operates under our familiar real number system, as such, I will not be discussing the Hyperreal decision framework that has been shown to resolve the problems and dilemma plaguing the Wager. Readers interested in knowing how the problems are resolved are advised to look into Frederik Herzberg's 2011 article titled *Hyperreal Expected Utilities and Pascal's Wager*¹.

1 Introduction

Pascal's Wager is a pragmatic argument that sought to establish the optimal course of action towards the Christian faith based on rational decision-making. It made no direct appeal to ontology, but rather employed a crude form of the modern-day theory of rational choice to inform belief in God. In other words, instead of taking the evidentialist approach to draw people to the Christian religion in line with the epistemic tradition of Pascal's time, Pascal argued that it is to the best interest of any rational agent to believe in the Christian God. This places Pascal's Wager firmly within the perimeter of decision theory and consequently requires a brief exposition of the decision-theoretic framework governing the argument.

One way to encapsulate the philosophical argument is to fit it into the state-of-the-world model for decision-making under uncertainty. The paradigm first stipulates a set of actions $A = \{a_1, a_2, \dots, a_m\}$, then with probability say p_j , if the states of the world $S = \{s_1, s_2, \dots, s_n\}$ are observed, there will be a reward r_{ij} generated for each action and state-of-the-world pair.² The decision-maker can then apply the decision criteria that best fits her risk profile to inform her action. The criterion most relevant to Pascal's Wager is the expected maximization of utility that identifies the optimal decision as the action that yields the greatest expected utility. For any

¹ Frederik Herzberg, "Hyperreal Expected Utilities and Pascal's Wager," *Logique et Analyse* 54, no. 213 (January, February, March 2011): 69-108, JSTOR.

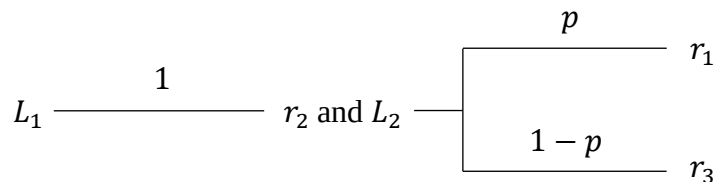
² Winston Wayne, "Decision Making under Uncertainty," in *Introduction To Probability Models*, 4th ed. (Belmont: Brooks/Cole, 2004), 37-40.

given lottery $L = (p_1, r_1; p_2, r_2; \dots; p_n, r_n)$, the expected utility is defined as the sum of the utility of the r_i 's ($u(r_i)$) weighted by their respective probability p_i 's, i.e.,

$$E(U_L) = \sum_{i=1}^n p_i u(r_i)$$

In the Wager, the set of actions can be construed as $A = \{\text{Wager for the Christian God}, \text{Wager against the Christian God}\}$ and the states of the world as $S = \{\text{God exists}, \text{God does not exist}\}$. The outcomes in Pascal's argument, on the other hand, are not the reward associated with each pair of action and state of the world itself, but a utility function of the rewards, i.e., $u(r_i)$. We do not know how exactly Pascal's utility function was defined, but it suffices to know that the utility associated with wagering for the Christian God when that very God exists is infinite while those associated with all other action and state-of-the-world pairs are finite if we toss out the consideration of the disutility of hell. A more detailed exposition on how Pascal's Wager is formalized through the decision matrix in relevant literature is given in the next section.

The utility theory used in Pascal's Wager is predicated on a set of axioms. Here, I give the definitions to only those that are violated in the presence of infinite utility. In terms of notation, I use p in between two lotteries (L_i) or rewards (r_j) to denote the object to the left of p is strictly preferred over the object to the right of p . For instance, $r_i p r_j$ means that I strictly prefer reward r_i over reward r_j . Similarly, I use i in place of p to denote equivalence. There are two axioms in the Von Neumann-Morgenstern utility theorem that are susceptible to infinite utility—the Continuity Axiom and the Unequal Probability Axiom. The former states that if $r_1 p r_2$ and $r_2 p r_3$, then there exists a probability p ($0 < p < 1$) such that $L_1 i L_2$, where



that is, the decision maker is indifferent between the certainty of getting reward r_2 and playing a lottery with probability of p of getting reward r_1 and $1 - p$ of getting reward r_3 . The Unequal Probability Axiom on the other hand states that given $r_1 p r_2$, if two lotteries have only r_1 and r_2 as their possible outcomes, it must be the case that the decision maker will prefer the lottery that yields the highest probability of obtaining reward r_1 .³ As we shall see, both axioms surrender to the force of infinite utility.

³ Ibid., 41-54.

2 The Decision Matrix

As acknowledged by most, if not all, discourse on Pascal's Wager, the argument is quite protean. Pascal, for one, never brought up the punishment of hell associated with disbelief when the Christian God exists in his *Pensée*; however, one can rightly incorporate the disutility of hell into the decision matrix if one holds a cynical view of humanity. For the sake of staying faithful to Pascal's ingenuity, the paper will assume that the utility of the above scenario is finite.

Pascal, in his rather scatter-brained treatise on the topic, began by introducing the notion of staking one life to gain multiple lives. In modern casino jargon, the payout would read something like 2 to 1 or 3 to 1. In Pascal's own words, "you would be imprudent, when you are forced to play, not to chance your life to gain three [lives] at a game where there is an equal risk of loss and gain."⁴ Observant readers might have already spotted a fatal flaw in the argument. There is no reason for the probability of winning and losing to be equal, i.e., each occurs with probability $\frac{1}{2}$, instead of an arbitrary probability $p > 0$. But of course, Pascal did not end there, his argument was precisely that "there is here an infinity of infinitely happy life to gain, a chance of gain against a finite number of chances of loss, and what you stake is finite."⁵ Without loss of generality then, the argument can be captured by the following decision matrix:

	God exists	God does not exist
Wager for God	∞	$f1$
Wager against God	$f2$	$f3$

where $f1$, $f2$, and $f3$ are finite utility values. It follows from the rule of expected maximization of utility described in the introduction that one should wager for God since for any positive probability p that God exists,

$$E(U_{Wager\ for\ God}) = p \cdot \infty + (1 - p)f1 = \infty > p \cdot f2 + (1 - p)f3 = E(U_{Wager\ against\ God})$$

As Bartha pointed out in his paper on the Wager, this particular formulation hinges on a critical premise—the probability p that God exists must be positive and finite.⁶ As we will see, this reliance on finite probability forms the basis for the Many Gods Objection. Before diving into the objections, let's first take a look at how conventional decision theory is beleaguered by infinite utility.

⁴ Blaise Pascal, *Pensée*, trans. John Warrington (London: Dent, 1932)

⁵ Ibid.

⁶ Paul Bartha, "Taking Stock of Infinite Value: Pascal's Wager and Relative Utilities," *Synthese* 154, no. 1 (January 2007): 6, JSTOR.

3 For Never Was a Story of More Woe Than the Demise of Decision Theory in the Face of Infinity

In the introduction, I put down two axioms of interest taken from the Von-Neumann Morgenstern utility theorem. Here, I address how they fail to stand their grounds in the face of infinite utility.

The first ill-fated axiom states that if $r_1 p r_2$ and $r_2 p r_3$, then there exists a probability p ($0 < p < 1$) such that the decision-maker is indifferent between the certainty of receiving reward r_2 and playing a lottery with probability p of getting r_1 and $1 - p$ of getting reward r_3 . Now, suppose r_2 and r_3 are any finite values such that $r_2 > r_3$, but $r_1 = \infty$. Since $r_1 > r_2 > r_3$, $r_1 p r_2$ and $r_2 p r_3$; the hypothesis is satisfied. However, any p ($0 < p < 1$) multiplied by infinity is infinity. The lottery with probability p of getting r_1 and $1 - p$ of getting reward r_3 will always be preferred over that of a certainty of receiving reward r_2 since the expected utility of the former is infinite, which always trumps the latter. It follows that for no probability p ($0 < p < 1$) is the decision maker indifferent between the two lotteries. The Continuity Axiom is therefore violated. Similar argument can be made against the Unequal Probability Axiom. The axiom states that given $r_1 p r_2$, if two lotteries have only r_1 and r_2 as their possible outcomes, then a decision maker will prefer the lottery that yields the highest probability of receiving r_1 . Now, suppose r_2 is any finite value, but $r_1 = \infty$, then $r_1 > r_2$, which implies that $r_1 p r_2$; the hypothesis is met. However, regardless of which lottery has a higher probability of obtaining r_1 , the expected utility of both lotteries will be the same; both are infinite. It follows that the decision maker is indifferent between the two lotteries, and there is no way a decision maker can prefer one lottery over another when the expected maximization of utility explicitly states that the two lotteries are equivalent.

It turns out that the problem plaguing the axioms of conventional decision theory is the very same one that plagues the wager itself. Hájek termed it “reflexivity under multiplication,” i.e., for any $p > 0$, $p \cdot \infty = \infty$.⁷ While “reflexivity under addition” is a desirable characteristic for the argument, its multiplication counterpart gives rise to many objections that cripple the Wager; among them the subversion of the underlying decision-theoretic framework we just witnessed, and objection raised on the basis of mixed strategies, which I will now turn to.

4 Mixed Strategies

⁷ Alan Hájek, “Waging War on Pascal’s Wager,” *The Philosophical Review* 112, no. 1 (January 2003): 46-48, JSTOR.

Outright wagering for or against God is not the only strategy we can play Pascal's game. One could, say, toss a coin and wager for God if it lands on head and wager against God if it lands on tail. This is an example of a mixed strategy, where decision makers choose an action based on a probability distribution as opposed to choosing an action outright (pure strategy). The use of mixed strategies highlights the detrimental effect the reflexivity property of infinity under multiplication has on the wager; suppose we roll a die and wager for God if the face value comes up to be four and wager against God otherwise, then the expected value associated with the strategy would be:

$$EV = \frac{1}{6}(p \cdot \infty + (1 - p) \cdot f1) + \frac{5}{6}(p \cdot (f2) + (1 - p) \cdot f3) = \infty$$

But one also obtains infinite expectation for choosing to wager for God if and only if, say, one is hit by a lightning, if and only if a comet strikes the Earth, if and only if Jesus returns tomorrow with his heavenly entourage to fulfill the messianic prophecy, etc. In fact, any action one performs has arguably a nonnegligible chance leading one to the belief in God. It follows that whatever one does, one invariably achieves maximal expectation. In that case, as Hájek pointed out, one might just as well practice devil worship and be perfectly justified for doing so since "nothing in [Pascal's] argument favors wagering for God over all of these alternative strategies."⁸ Having seen decision theory yielded to the belligerence of infinite utility and the Wager sabotaged by the reflexive property of infinity under multiplication, it is time to question the assumption that has been held true up to this point, namely that the probability of God's existence is positive and finite.

5 The Many Gods Objection

The Many Gods Objection, as its name implies, is raised on the ground of the presence of many (potentially infinite) different conceptions of God. In Saka's treatment on the Wager, Ellen DeGeneres was cited to have joked about the possibility of God being a giant bug that punishes those who mistreat its earthly counterparts.⁹ Voltaire's God, as another example, is essentially a "clockmaker who set the world in motion and then stood back to watch it tick[s];"¹⁰ a noninterventionist God that has no interest in earthly affairs. One could easily invent Gods by adding to or subtracting from existing attributes, thereby creating an infinite number of candidate Gods that are equiprobable in terms of their existences. Pascal's God then, would just be one in

⁸ Ibid., 31.

⁹ Ellen DeGeneres, *My Point...and I Do Have One* (New York: Bantam Doubleday, 1995). 129, in Paul Saka, "Pascal's Wager and the Many Gods Objection," *Religious Studies* 37 (2001).

¹⁰ CrashCourse, "Candide: Crash Course Literature 405," YouTube Video, 12:12, December 13, 2017.

an infinite number of potential Gods. This, of course, poses a threat to the calculation of expected utility, for given that the probability of Pascal's God exists $p = \frac{1}{\infty}$,

$$E(U_{Wager\ for\ God}) = \frac{1}{\infty} \cdot \infty + \left(1 - \frac{1}{\infty}\right) \cdot f1 \cong \frac{\infty}{\infty} + f1 = ?$$

Since infinity over infinity is undefined, the expected utility for wagering for God is indeterminant.

Pascal seems to have taken this into account when he employed the rather weird notion of life as units of payoff/utility: "if there were an infinity of chances, of which one only would be for you, you would still be right in wagering one [life] to win two [lives]."¹¹ Pascal was clearly operating under the assumption that the utility for wagering for God trumps that of wagering against God when God doesn't exist. This need not be the case however, and one may even argue otherwise if one finds consolation in earthly pleasure as opposed to the Christian faith. Our generalized formulation does not make such assumption, and consequently we find ourselves at a crossroad when playing Pascal's 2 to 1 payoff game, for the expected utility of each action now becomes

$$E(U_{Wager\ for\ God}) = \frac{1}{\infty} \cdot 2 + \left(1 - \frac{1}{\infty}\right) \cdot f1 = \frac{2}{\infty} + f1 \cong f1$$

$$E(U_{Wager\ against\ God}) = \frac{1}{\infty} \cdot f2 + \left(1 - \frac{1}{\infty}\right) \cdot f3 = \frac{f2}{\infty} + f3 \cong f3$$

There is no guarantee that $f1 > f3$, and we therefore enter a state of indeterminacy. Moreover, viewing utility as finite life units undermines the tenet of Pascal's argument. Remember, the crux of the argument is predicated on the fact that the utility of wagering for God when God exists is so large (potentially infinite) such that we need not worry about the probability of God's existence and the exact utilities of other action and state-of-the-world pairs; this is what Hájek termed the "Requirement of Overriding Utility," which I will explore further in the next section.

6 Requirements for Reformulation

In order to address the problems associated with the Wager that I have been trumpeting up to this point and to stay philosophically faithful to Pascal's argument, Hájek – in his treatise on the topic – set forth two requirements to be taken into account for any attempt at reformulating the Wager:

¹¹ Pascal, *Pensée*, trans. Warrington.

Requirement of Overriding Utility: we alluded to this in the last section. The objective is to have the utility of salvation be sufficiently large so as to overwhelm “any other utilities that enter into the expected utility calculations,” thereby “rendering irrelevant the exact value of the probability one assigns to God’s existence.”¹²

Requirement of Distinguishable Expectations: The rationale for having this condition is to address the problem of mixed strategies. The goal is to be able to differentiate the expectation of strategies having different probability distribution so that a strategy that has a higher probability of leading to belief in God should correspond a higher expectation and vice versa.¹³

To satisfy the former, the utility of salvation must be reflexive under addition by any positive real number and multiplication by any positive real number greater than one, that is, $\infty + x = \infty$ for any $x \in \mathbb{R}^+$ and $\infty \cdot x = \infty$ for any $\{x \in \mathbb{R}^+ | x > 1\}$. This guarantees that the outcome of wagering for God when God exists is the greatest imaginable payoff. To satisfy the latter, the utility of salvation must not be reflexive under multiplication by any positive probability p , that is, $p \cdot \infty \neq \infty$ for any p ($0 < p < 1$) since being reflexive in that case opens up the flood gate of mixed strategies. I now present one of Hájek’s reformulation of the Wager.

7 Finite Utility to the Rescue: Hájek’s Reformulation

We have now seen that infinite utility is trouble. Up to this point, our discussion has been fixing the utility of wagering for God when God exists (i.e., the utility of salvation) to infinity and toying with trivial mathematical technicalities to discredit the Wager. My operating assumption was of course that the argument was meant to appeal to theists, atheists and anyone in between the extremes alike. Hájek suggested however, that we could instead shift the variable that is held constant to a set of target audience that would assign a positive, finite probability to God’s existence and “solve” for the utilities instead. Let us assume that there exists a set S that contains people that existed and will exist, who would assign a positive and finite probability to God’s existence; consequently, this set will not contain any atheist or agnostic whose probability for the said event would be 0 or infinitesimal. By eliminating infinitesimal probability, we have conveniently thrown out the bathwater (i.e., the concern over the Many Gods Objection) without throwing out the baby (the Wager’s intended audience), for – as Hájek pointed out – those at the disbelief end of the belief spectrum are unlikely to be convinced by the Wager anyway.¹⁴ Now, large as it will become, the set S will not be infinite since the human race is unlikely to survive a

¹² Hájek, “Waging War,” 34.

¹³ Ibid.

¹⁴ Ibid., 43-45.

cataclysmic event similar to the one that destroyed our reptile counterparts 65 million years ago; it is therefore fair to consider the smallest probability that this set of skeptics and diehard theists assign to God's existence. This probability, to steal Hájek's notation p_{min} , represents the most ferocious of skeptics. To convince this overly suspicious individual, Pascal needs only to fix this person's probability and solve for $f_{U(salvation)}$ from the inequality,

$$p_{min} \cdot f_{U(salvation)} + (1 - p_{min}) \cdot f_1 > p_{min} \cdot f_2 + (1 - p_{min}) \cdot f_3$$

with variables from the following decision matrix:

	God exists	God does not exist
Wager for God	$f_{U(salvation)}$	f_1
Wager against God	f_2	f_3

Let us now see if the reformulation that utilizes the finite representation of utility passes Hájek's own litmus test. Take $q > 0$ to be any probability for wagering for God derived from a specific probability distribution, then the Requirement for Distinguishable Expectation is met since

$$EV = q \cdot (p_{min} \cdot f_{U(salvation)} + (1 - p_{min})f_1) + (1 - q) \cdot (p_{min} \cdot f_2 + (1 - p_{min})f_3)$$

is now finite and consequently different from the expectation of outright wagering for God and, as Hájek pointed out, the expectation is in fact a strictly increasing function of q ; a higher probability of leading to belief in God then, would yield a higher expectation, as desired; the mixed strategies obstacle is thereby surpassed. Moving on to the Requirement of Overriding Utility, since the value of $f_{U(salvation)}$ was chosen deliberately to swamp any finite utilities in the decision matrix, it is unquestionably large enough to render any positive and finite probability assigned to God's existence irrelevant. But herein lies a caveat; previously, we stated that the degree of compliance to the Requirement of Overriding Utility a reformulation attains is assessed by the reflexivity of the utility of salvation under addition by any positive reals and multiplication by any positive reals greater than 1. Our finite utility representation of salvation surrenders to this condition since $f_{U(salvation)} + 1$ is strictly greater than $f_{U(salvation)}$, and so is $f_{U(salvation)} \cdot 2$, say; our finite utility for salvation is very far from being the greatest imaginable reward. Hájek's dilemma has hereby entered the spotlight.

8 Hájek's Dilemma

In modelling Pascal's Wager, using infinite utility to represent salvation runs into the problem of mixed strategies due to the reflexive nature of infinity under both addition and multiplication by positive real numbers. Finite utility representation of salvation, on the other hand, is immune to the mixed-strategies line of attack, but its faithfulness to Pascal's argument is severely compromised as it turns the infinity of heavenly bliss into a terminable finitude as a result of the irreflexive nature of finite numbers. The solution then, is easier said than done: the utility of salvation must be reflexive under addition by any positive real numbers and multiplication by any positive reals greater than 1 to ensure that Pascal's "infinity of an infinitely happy life" is respected, but the same utility must also not be reflexive under multiplication by positive probabilities so that mixed strategies do not knock off the dominoes of maximal expectations. In other words, the utility of salvation must exhibit both the attributes of infinity and finitude. But how can this be achieved? Hájek asked; how can the utility of salvation exist in duality, being both infinite and finite at the same time? The question is throwing bones at dog. It seems the elixir to the problem we are searching for, as Hájek cynically implies, does not exist.

9 Conclusion

Pascal's Wager – albeit an ingenious attempt at justifying religious belief – was theoretically constructed on shaky grounds. Not only does its underlying decision-theoretic framework falter at the use of infinite utility, mixed strategies and the Many Gods Objection are also able to relegate the argument to the realm of sophistry. Genuine reformulations of the Wager have been proposed, but their usefulness was circumvented by Hájek's dilemma, where in staying truthful to Pascal's argument, the Requirement of Distinguishable Expectation must of necessity be relinquished, and in addressing the objection raised on the basis of mixed strategies, the faithfulness to the author's intents must be compromised. Wherein lies the solution to the problem then? One must perhaps look beyond the reals.

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