The Black-Scholes Option Pricing Model

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Abstract

This paper aims to introduce the basic concept of the Black-Scholes option pricing model and explore the implications of its limitations. First, we will discuss some of the most important options basics and put-call parity to enable in order to further explain the model. Then we will elaborate the underlying assumptions of the model and use examples to show how to apply the model in real-world. Lastly, we will discuss the limitations of the model and its implications to to financial mathematical modeling.

1 Introduction and Historical Background

The Black-Scholes-Merton model, sometimes just called the Black-Scholes model, is a mathematical model of financial derivative markets from which the Black-Scholes formula can be derived. This formula estimates the prices of call and put options. Originally, it is used to price European options and was the first widely adopted mathematical formula for pricing options. Some credit this model for the significant increase in options trading. And it has a significant influence in modern financial pricing. Prior to the invention of this formula and model, options traders didn't all use a consistent mathematical way to value options, and empirical analysis has shown that price estimates produced by this formula are close to observed prices.[5]

The Black-Scholes model was developed in 1973 by Fischer Black, Robert Merton, and Myron Scholes and is still widely used widely used by option traders today. In their initial formulation of the model, Fischer Black and and Myron Scholes, the economists who originally formulated the model, came up with a partial differential equation known as the Black-Scholes equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

where for European call or put option on an underlying stock of price S at time t, paying no dividends, the price V of the option as a function of S as shown above (r is risk-free interest rate and σ is the volatility of the stock).

Later Robert Merton published a mathematical understanding of their model, using stochastic calculus that helped to formulate what became known as the Black-Scholes-Merton formula. Both Myron Scholes and Robert Merton split the 1997 Nobel Prize in Economists, listing Fischer Black as a contributor, though he was ineligible for the prize as he had passed away before it was awarded.

The formula helped to legitimize options trading, making option trading less like gambling and more like science. Today, the Black-Scholes-Merton formula is widely used, though in individually modified ways, by traders and investors, as it is the fundamental strategy of hedging to best control, or mitigate, risks associated with volatility in the assets that underlie the option.[2]

2 Option Basics

In finance, an option is a contract which grants its owner, the holder, the right, but not the obligation, to buy or sell an underlying asset or instrument at a specified strike price prior to or on a specified date, depending on the form of the option. In fact, contracts similar to options are believed to have been used since ancient times. In London, puts and "refusals", which is similar to call options first became well-known trading instruments in the 1690s during the reign of William and Mary. Privileges were options sold over the counter in nineteenth century America, with both puts and calls on shares offered by specialized dealers.

Options are powerful because they can enhance an individual's portfolio through added income, protection, and even leverage. Depending on the situation, there is usually an option scenario appropriate for an investor's goal. A popular example would be using options as an effective hedge against a declining stock market to limit downside losses, which is also known as protective put (*Figure 1*). Options can also be used to generate recurring income using strategies such as covered call or covered put.

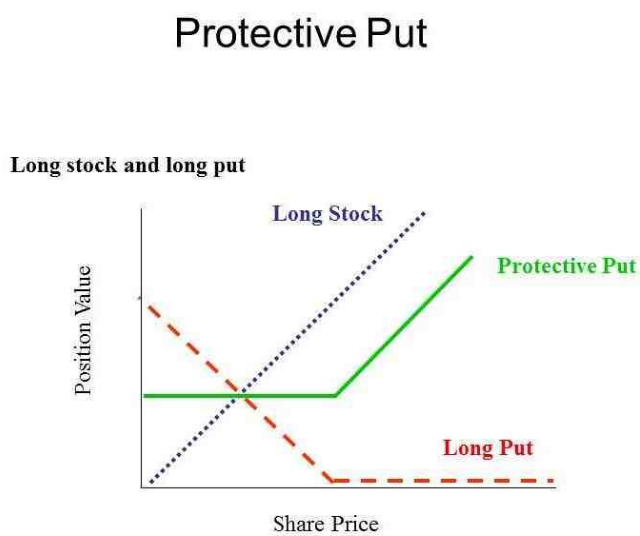

Figure 1: An example of option trading strategy that protects the investors with long position from downside losses against a declining stock market

As metioned above, an options contract offers the buyer the opportunity to buy or sell, depending on the type of contract they hold, the underlying asset. Unlike futures, the holder is not required to buy or sell the asset if they choose not to. There are two types of options:

- **Call options** allow the holder to buy the asset at a stated price within a specific time frame.
- **Put options** allow the holder to sell the asset at a stated price within a specific time frame.

Options contracts usually represent 100 shares of the underlying security, and the buyer will pay a premium fee for each contract. For example, if an option has a premium of 25 cents per contract, buying one option would cost 25 dollars ($0.25 \times 100 = 25$). Each option contract will have a specific **expiration date**, often referred to as the **expiry**, by which the holder must exercise their option. The stated price on an option is known as the **strike price**. In other words, strike price is the price for buying or selling the security until the expiration date. Options are typically bought and sold through online or retail brokers and the price paid by the option buyer is called an option premium.

Moreover, there are two different option styles that are currently offered on the market, which are American options and European options. American and European options have similar characteristics but the differences are important. For instance, owners of American-style options may exercise at any time before the option expires. On the other hand, major broad-based indices, including the S&P 500, have very actively traded European-style options, while owners of European-style options may exercise only at expiration. Note that the standard Black-Scholes model is only used for calculating prices for European-style options and does not take into account that American-style options could be exercised before the expiration date.

There are some jargon that option traders often use to describe the relationships between market price and strike price. An **in-the-money (ITM)** option is one with a strike price that has already been surpassed by the current stock price. An **out-of-the-money (OTM)** option is one that has a strike price that the underlying security has yet to reach, meaning the option has no intrinsic value. An **at-the-money (ATM** option is one with a strike price that is equal to the current stock price. In options trading, the difference between "in the money" and "out of the money" is a matter of the strike price's position relative to the market value of the underlying stock, called its **moneyness**. Since ITM options have intrinsic value and OTM options do not, ITM options carry a higher premium than OTM options. [5] However, in the money or out of the money options both have their pros and cons. One is not better than the other. Rather, the various strike prices in an options chain accommodate all types of traders and option strategies. The bottome line is, when it comes to buying options that are ITM or OTM, the choice depends on your outlook for the underlying security, financial situation, and what you are trying to achieve.

3 Put-Call Parity for European Options

The Black-Scholes model can only be used to calculate the price of an European call option. In order to calculate the price of an European put option, we need to define the relationship between call price and put price of an European option. In financial mathematics, put–call parity defines a relationship between the price of a European call option and European put option, both with the identical strike price and expiry. [4] For this paper, we do not concern the derivation of put-call parity and will not discuss the mathematics behind this parity. However, it is important to understand the underlying principle of put-call parity. Put call parity states that holding up of the long European call with the short European put simultaneously will yield out the same return when you will be holding up a forward contract having the identical basic asset, as well as the expiry date. And here the forward price will be equivalent to the option's strike amount. This relationship can be demonstrated by the put-call parity formula:

$$C + \frac{K}{(1+r)^t} = S_t + P \quad \text{or} \quad C + Ke^{-rt} = S_t + P$$

where:

$$\begin{aligned} C &= \text{the European call options price} \\ K &= \text{the strike price} \\ S_t &= \text{the present market value of the underlying asset} \\ P &= \text{the European put option price} \\ r &= \text{the risk-free interest rate} \\ t &= \text{time until option expiration} \end{aligned}$$

The term $\frac{K}{(1+r)^t}$ (also can be written as Ke^{-rt}) is the present value of the strike price. Present value

(PV) is the current value of a future sum of money or stream of cash flows given a specified rate of return. Future cash flows are discounted at the discount rate, and the higher the discount rate, the lower the present value of the future cash flows. This is also known as the time value of money, which is the concept that money you have now is worth more than the identical sum in the future due to its potential earning capacity. This core principle of finance holds that provided money can earn interest, any amount of money is worth more the sooner it is received.

Recall that put-call parity states that simultaneously holding a short European put and long European call of the same class will deliver the same return as holding one forward contract on the same underlying asset, with the same expiration, and a forward price equal to the option's strike price. This is because if the price at expiry is above the strike price, the call (ITM call) will be exercised, while if it is below, the put (ITM put) will be exercised, and thus in either case one unit of the asset will be purchased for the strike price, exactly as in a forward contract.

Now if we rearrange the equation, the price of an European put option can be obtained using the formula:

$$P = C + \frac{K}{(1+r)^t} - S_t$$

4 The Black-Scholes Formula

The Black Scholes call option formula is calculated by multiplying the stock price by the cumulative standard normal probability distribution function. Thereafter, the net present value (NPV) of the strike price multiplied by the cumulative standard normal distribution is subtracted from the resulting value of the previous calculation.[3]

In mathematical notation, the Black-Scholes call option formula is given as following:

$$C = N(d_1)S_t - N(d_2)Ke^{-rt}$$

$$\text{where} \quad d_1 = \frac{\ln\frac{S_t}{K} + \left(r + \frac{\sigma^2}{2}\right)t}{\sigma\sqrt{t}}$$

$$\text{and} \quad d_2 = d_1 - \sigma\sqrt{t}$$

where

$$\begin{aligned}
C &= \text{Call option price} \\
S_t &= \text{Current stock price} \\
K &= \text{Strike price} \\
t &= \text{Time till expiration date} \\
r &= \text{Risk-free interest rate} \\
\sigma &= \text{Volatility or standard deviation of log returns on the underlying stock} \\
N(d_1)\&N(d_2) &= \text{Cumulative distribution functions for standard normally distributed random variables } d_1, d_2
\end{aligned}$$

The risk-free rate is usually equivalent to the rate of return an investor could get on an investment assumed to be risk-free like a Treasury bill.

5 Underlying Assumptions of the Black-Scholes Model

Before we use any mathematical models in solving real-world problems, it is important to learn about the model's limitations. In order to do so, we need to understand the assumptions lie beneath the model. Just like any other mathematical models, the Black-Scholes model makes certain assumptions. In this section, we will discuss several important assumptions of the Black-Scholes model.

5.1 Lognormal Distribution on stock prices

The Black-Scholes model assumes that stock prices follow a lognormal distribution based on the principle that asset prices cannot take a negative value; they are bounded by zero. This is also known as a Gaussian distribution. Often, asset prices are observed to have significant right skewness and some degree of kurtosis, also known as fat tails. This means high-risk downward moves happen more often in the market than a normal distribution predicts.

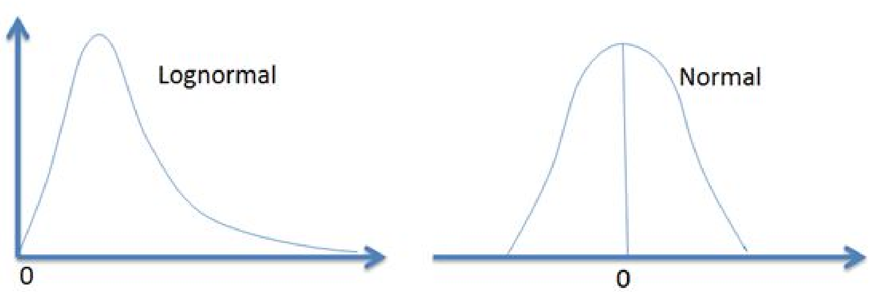

Figure 2: Lognormal distibution vs. Normal distribution

5.2 No Dividends

The Black-Scholes model assumes that the underlying stocks do not pay any dividends or returns. In other words, no dividends are paid out during the life of the option. As previously mentioned, put-call parity states that simultaneously holding a short European put and long European call of the same class will deliver the same return as holding one forward contract on the same underlying asset. With dividends, estimating the forward price of the underlying at exercise date would become troublesome. However, there are extensions of the Black-Scholes models which include dividends, but for the original model presented in this paper, there will be no dividends paid.

5.3 Expiration date

The model assumes that the options can only be exercised on its expiration or maturity date. Hence, it does not accurately price American options. It is extensively used in the European options market. Similar to the no dividends case, there are extensions of the Black-Scholes models which can calculate the theoretical price of American options. We will mentioned such extensions in the section where the implications of the model is discussed.

5.4 Random walk

The stock market is highly volatile, and therefore a state of random walk is assumed as the market direction can never truly be predicted (*Figure 3*). So what exactly is a random walk? A "random walk" is a statistical phenomenon where a variable follows no discernible trend and moves seemingly at random. The random walk theory, as applied to trading, most clearly laid out by Burton Malkiel, an economics professor at Princeton University, posits that the price of securities moves randomly. Therefore, any attempt to predict future price movement, either through fundamental or technical analysis, is futile.[] This is also known as the efficient market hypothesis (EMH), which states that asset prices reflect all available information. A direct implication is that it is impossible to "beat the market" consistently on a risk-adjusted basis since market prices should only react to new information.

Figure 3: A "random walk" in the stock market

5.5 Frictionless market

The model assumes costless trading, i.e. no transaction costs, including commission, brokerage, and liquidity risks.

5.6 Constant risk-free interest rate

The risk-free interest rates is assumed to be constant over the option duration. In other words, the model assumes that the rate of return of a risk-free investment such as a Treasury bill will remain constant during the life of an option.

5.7 Normal Distribution on stock returns

Stock returns are assumed to be normally distributed, which implies that the volatility of the market is constant over time.

5.8 No arbitrage opportunities

The model assumes there is no arbitrage opportunities, which means put-call parity always holds true. This assumption ensure that all the markets have matching prices on identical or similar financial instruments in different markets or in different forms.

6 Applying the Black-Scholes Model

In this section, we will demonstrate how to apply the Black-Scholes model using a simple example and discuss the difference between historical volatility and implied volatility.

Now, suppose we want to value a TSLA NOV 1,2020 100 call, i.e. the strike price on a call option on Tesla stock that expires on November 1st is \$100. Tesla closed at \$117.25 on August 1 (92 days before option expiration). Now we need the risk-free interest rate and the stock volatility to value the call. One of the most convenient way is to consult the "Money Rate" section of the Wall Street Journal. As shown in *Figure 4*, we we find a T-bill rate with about 92 days (13 weeks) to maturity to be 8.5%.

Treasury Bill Auction

			52-WEEK	
	LATEST	WK AGO	HIGH	LOW
4 weeks	**0.080**	0.085	1.620	0.000
13 weeks	**0.085**	0.090	1.560	0.000
26 weeks	**0.090**	0.100	1.580	0.080

Figure 4: The risk-free interest rate for Treasury bill with different maturities

To determine the volatility of returns, we need to take the logarithm of returns and determine their volatility. To simply the process for explanatory purpose, we can find the 90-day historical volatility (close-to-close) online, which is 0.8445. Note that close-to close means the past volatility of the security over the selected time frame, calculated using the closing price on each trading day.

So we have all the required data to calculate the call price. Recall the Black-Scholes formula:

$$C = N(d_1)S_t - N(d_2)Ke^{-rt}$$

$$\text{where} \quad d_1 = \frac{\ln \frac{S_t}{K} + \left(r + \frac{\sigma^2}{2}\right) t}{\sigma\sqrt{t}}$$

$$\text{and} \quad d_2 = d_1 - \sigma\sqrt{t}$$

Now we have

$$
\begin{aligned}
S_t &= \$117.25 \\
K &= \$100 \\
t &= 92 \text{ days (time to maturity is usually expressed in years, thus, } \frac{92}{365} = 0.2520) \\
r &= 8.5\% \quad \text{or} \quad 0.085 \\
\sigma &= 0.8445 \\
d_1 &= \frac{\ln \frac{117.25}{100} + \left(0.085 + \frac{0.8445^2}{2}\right) 0.2520}{0.8445\sqrt{0.2520}} = 0.6369 \\
d_2 &= 0.6369 - 0.8445\sqrt{0.2520} = 0.2109
\end{aligned}
$$

Now we just need to plug d_1, d_2 into software such as Excel to calculate the cumulative distribution of d_1 and d_2 using function NORMSDIST(). The results come out to be:

$$N(d_1) = 0.728 \quad \text{and} \quad N(d_2) = 0.584$$

If we put everything together using the Black-Scholes formula, we have:

$$C = 0.728 \times 0.2520 - 100e^{-0.2520 \times 0.085} 0.584 = 29.4$$

So the theoretical price of this Tesla call option is approximately \$29.4. Now if we want to calculate the price of a Tesla put option of the same class, we can use put-call parity formula:

$$P = 29.4 + \frac{100}{(1 + 0.085)^{0.2520}} - 117.25 = 10.03$$

However, in order to validate our estimation, we observed that such Tesla call option is actually sold for 27.91. The only thing that could be wrong in our calculation is the volatility estimate. This is because we need the volatility estimate over the option's life, which we cannot observe. So now we have the actual call price, we can calculate the implied volatility, which gives an estimate of what the market thinks about likely volatility in the future.

If we substitute everything back into the formula, including the observed call price, and solve for volatility, we will get the implied volatility. In this case, the implied volatility for Tesla over this 92-day period is approximately 0.7703, which is lower than the historical volatility we used to value to call option (0.8445). As we can see in the example, the Black Shcoles model cannot always price the option accurately since the model is based on a few assumptions that are simply unrealistic in the real stock market. There are many factors that were not accounted for by the model. Therefore, we have to pay close attention to the limitations of the Black-Scholes model.

7 Limitations of the Black-Scholes Model

The assumptions described previously also impose limitations on the Black-Scholes model in a number of ways. First of all, the Black-Scholes model is limited to the European option market. As stated previously, the model is only used to price European options and does not take into account that American options could be exercised before the expiration date. Most options traded today are American call options that can be sold at any point. American options are the options that most individual investors are acquainted with. If you trade options on stocks, like Apple (AAPL), General Electric (GE) or Google (GOOG), you are most likely trading American-style options. On the other hand, European-style options are typically less familiar to retail traders because they are much less common. The options on indexes, like the S&P 500, the NASDAQ Index or the Russell 2000 Index, or on currency pairs (Forex), are usually European-styled. However, there are extensions of the model that can be used to calculate the price of an American option.

The problem of finding the price of an American option is related to the optimal stopping problem of finding the time to execute the option. Since the American option can be exercised at any time before the expiration date, the aforementioned Black–Scholes equation becomes a variational inequality of the form:

$$\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV \leq 0$$

together with $V(S,t) \geq H(S)$ where $H(S)$ denotes the payoff at stock price S and the terminal condition: $V(S,t) = H(S)$.[1] In general, this inequality does not have a closed form solution like the Black-Scholes formula. Investors and traders have to derive this equation according to the options they are dealing with. Thus, in order to calculate the price of an American opiton, we would have to go through some grunt mathematical derivations and it would not be as convenient as calculating the price of an European option.

Secondly, the model assumes the risk-free rates are constant, but in reality, this hardly ever holds true. The risk-free rates constantly fluctuate. The risk-free rates might not act as volatile as the stock prices, ignoring small changes in the risk-free rates might still cause option traders to over- or under-price an option and eventually leads to losses. Moreover, volatility is also assumed constant over the option's life. As shown in the example, the implied volatility was almost 8% lower than the historical volatility used to calculated the call price, which resulted in a spread between the theoretical price and the market price of the call option.

Moreover, the model assumes costless and continuous trading. In real-world scenario, trading generally comes with exchange fees, the costs to buy or sell stocks and options, and the cost of time; the time it takes for the order to go through may result in changes to the price on the market. These costs can be managed but are not included in the model. Also the model assumes that trading occurs continuously, unlike reality, where markets shut down for the night and then can reopen at significantly different prices to reflect new information.

Another limitation we need to be aware of is that the stock market does not always follow the random walk hypothesis. In real world, we frequently observe large price swings in the stock market, which makes it possible to beat the market in a period of time. The model ignores the occurrence of these price swings, which could cause troubles for option traders when such price swing happens.

There are a few other limitations that are caused by operational issues, include assuming no penalty or margin requirements for short sales, no arbitrage opportunities and no taxes. In reality, all these do not hold true; either additional capital is needed or realistic profit potential is decreased.

8 Implication to Quantitative Financial Modeling

Mathematics helped financial instrument pricing estimations become more and more precise via the development of models like the Black-Scholes model. Fields in mathematics such as statistics and probability serve as the backbone of financial quantitative modeling and provide investors a more insightful look at the operating mechanism of the financial market. However, blindly following any mathematical or quantitative trading model may lead to uncontrolled risk exposure. As mentioned before, financial failures of 2008–09 are attributed to the flawed use of trading models. Despite the challenges, mathematical or quantitative model-based trading continues to gain momentum. Complex trading instruments such as derivatives continue to gain popularity, as do the underlying mathematical models of valuation. While no model is perfect, being aware of limitations can help in making informed trading decisions, rejecting outlier cases and avoiding costly mistakes that may result in huge losses. A cautious approach with clear insights about the limitations of a model, their repercussions, available alternatives, and remedial actions can lead to safe and profitable trading.

9 Conclusion and Future Research

The Black-Scholes model is undoubtedly one of the most important financial models in derivative trading as it theorizes option trading and gives the investors more confidence in the derivative market. The Black-Scholes model helped to create the now multi-trillion dollar derivatives market. However, while no mathematical model is perfect, the Black-Scholes model has its limitations. Several assumptions of the model do not hold true in real financial markets. Investors and traders often overlook the assumptions made in the derivation of the model and apply it abusively to the real markets.

Nonetheless, the model itself is not the real problem. It is useful and precise and its limitations are clearly stated. It provides an industry-standard method to assess the theoretical value of a financial derivative, so the derivative can be traded before maturity. The Black-Scholes formula is accurate if investors use it sensibly and are willing to seek an alternative pricing method when the market conditions are not appropriate. The moral of the story is that any quantitative financial models have their caveats. Before we apply them to solve real world problems, we must not forget to ask how reliable the answers would be if market conditions changed.

In future research, we could further explore the derivation and extensions of the Black-Scholes model. A detailed look at how to modify the model to accommodate different market scenarios would be an interesting topic to research on. Moreover, perhaps we can discuss how to utilize machine learning to teach computers to price the financial derivatives like options using the most fitting method.

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