Mathematics and Navigation

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May 2024

Abstract

In this paper I briefly discuss the connection between mathematics and navigation. I begin by outlining the history of navigation as well as its significance and challenges. I then explain key concepts from geography and astronomy to acquaint the reader with a basic knowledge of celestial navigation. Next I explore two navigational problems: Finding latitude and finding longitude. In this, I discuss the history of instruments used to determine latitude and longitude.

1 Introduction & Brief History

Until the advent of modern navigational technologies such as the Global Positioning System (GPS), the problem of navigation has been an open and challenging question for humans for much of history. Indeed, across many cultures, the question of one's location and of how to travel to another location—especially at sea, where the only reference points often are the sun, the moon, and other stars—has motivated human ingenuity to arrive at many clever techniques to determine position. Most of these techniques have required, in some form or another, mathematical knowledge; in particular, principles of geometry and trigonometry, many of which were known to ancient cultures such as Ancient Greece,[1] have been essential to the field of navigation. Furthermore, as mathematical knowledge broadened and technologies like the magnetic compass and the sextant were developed, navigational methods have improved, granting navigators (especially those at sea) both more accuracy and

[1] Marvin Jay Greenberg, *Euclidean and Non-Euclidean Geometries*: 4$^{\text{th}}$ ed. (W.H. Freeman and Company, 2008), 1-9.

precision in finding location. Today, of course, our mathematical knowledge and technology has improved such that the average person, untrained in any navigational art, can easily find their position within a latitudinal and longitudinal second.

But modern navigation stands on the shoulders of many mathematicians and navigators. Aside from the general geometrical and trigonometric knowledge inherited from ancient cultures, more modern mathematicians in the sixteenth through twentieth centuries have made significant contributions to navigation and, more specifically, cartography. For instance, Edward Wright, the English mathematician and cartographer born in 1561, was important for the making of an accurate map based on the Mercator projection[2] first developed by Gerardus Mercator in 1569.[3] The eminent mathematicians Johann Heinrich Lambert[4] and Leonhard Euler[5] were behind substantial mathematical theory behind principles of map projections and spherical trigonometry. From the late eighteenth and early nineteenth centuries, Nathaniel Bowditch was a skilled mathematician who simplified formulas for computing lunar distance (used for determining longitude)[6] and wrote the important text the *American Practical Navigator*, which is still in publication by the National Geospatial Intelligence Agency after many editions. More recently, Gladys West, the African American mathematician working in the late twentieth century, helped develop satellite navigation through her work on GPS.[7] In short, mathematicians have closely followed the problem of navigation, and there is substantial overlap between the two fields.

Besides navigation's mathematical appeal, navigation has been of significant cultural, political, and economic import. Any successful cultural and material exchange between peoples separated by an appreciable geographic distance relies on accurate navigation. Although this has been an important feature of most of human history, the impact of navigation became

[2] Frank J. Swetz, "Mathematical Treasure: Edward Wright's Certaine Errors in Navigation," *Convergence* (August 2021).

[3] "Mercator projection," *Encyclopedia Britannica*, November 26, 2023, https://www.britannica.com/science/Mercator-projection.

[4] John P. Snyder, *Flattening the Earth: Two Thousand Years of Map Projections* (University of Chicago Press, 1993), 77.

[5] See, for instance, https://mathnmaps.com/euler/index.html for a list compiling papers by Euler on the subject of cartography or geodesy.

[6] Nathaniel Bowditch, *American Practical Navigator: An Epitome of Navigation* (National Geospatial-Intelligence Agency, 2024), iii.

[7] "Gladys B. West," Virginia Changemakers, *Library of Virginia*, accessed May 11, 2024, https://edu.lva.virginia.gov/changemakers/items/show/362.

especially clear beginning in the Age of Discovery in the fifteenth and sixteenth centuries, as faring long distances at sea became increasingly common and necessary for international trade and the political organization of large empires and European colonial entities separated from Europe by the Atlantic.[8] And today's global trade network that connects economies and cultures that are physically separated by vast oceans, navigation has never been more crucial to human endeavors. Keeping in mind the historical importance of navigation, as well as the field's status as a mathematical art and science, let us attend to the exact details of some basic navigational principles: namely, latitude and longitude.

2 Some Geographic and Astronomic Concepts

The Earth is an oblate spheroid, meaning that its shape is roughly spherical but is wider at the equator than it is through the poles.[9] The Earth may also be generally modelled as an ellipsoid. Immediately, this creates difficulty and imprecision when we wish to construct the spherical coordinate grid of latitude and longitude that we use today. There are, in fact, several kinds of latitude and longitude such as astronomic, geodetic, and geocentric, which account for the imperfections of the Earth's roughly ellipsoid shape.[10] For the sake of simplicity, in this paper we will assume that the Earth is perfectly spherical, and that coordinates of latitude and longitude correspond to spherical coordinates.

Coordinates of latitude and longitude have ancient roots. The notion of a grid of lines of latitude of longitude on the Earth's surface is at least as old as the geographer Eratosthenes in the third century B.C.,[11] the same Eratosthenes who accurately measured the Earth's circumference.[12] That is to say, as early as ancient Greece, people had decent knowledge of the geography of the globe.

The coordinate system used today involves 180 degrees of latitude and 360 degrees of longitude. How are these measures calculated? Imagine that the Earth is embedded in a

[8] J. Brown Mitchell, "European exploration," *Encyclopedia Britannica*, May 9, 2024, https://www.britannica.com/topic/European-exploration.
[9] Bowditch, *Navigator*, 6.
[10] Ibid., 21.
[11] Duane W. Roller, *Eratosthenes' Geography* (Princeton University Press, 2010), 25–26.
[12] Ibid., 23.

three-dimensional Euclidean space, with the vertical axis coinciding with the line through the Earth's two poles, i.e., its axis of rotation. Then we may define the equatorial plane to be the plane that intersects the Earth's center and is perpendicular to the Earth's axis of rotation.[13] We define the latitude by the angle above or below the equatorial plane (and not the angle from the vertical axis). For example, the latitude of the Equator, which is the intersection of the equatorial plane and the Earth, is 0 degrees. Rather than designating the position of the angle as above or below the equatorial plane as a positive or negative angle, we designate angles above the equatorial plane by degrees North, and angles below it by degrees South. Then the latitude of the North Pole is 90 degrees North, and that of the South Pole is 90 degrees South.

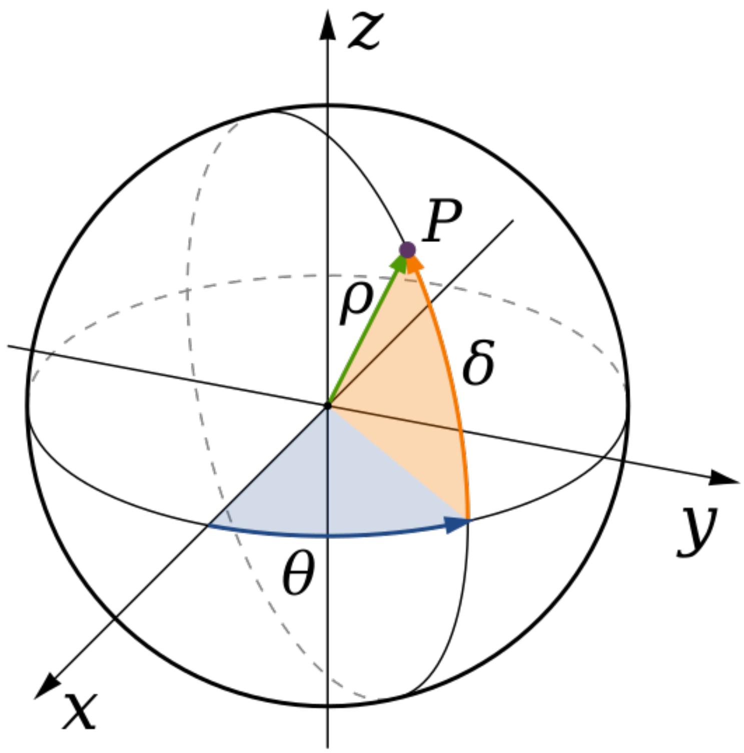
Lines of latitude are called parallels, and these are the intersections of the Earth and planes parallel (in the Euclidean sense) to the equatorial plane. Here it may be helpful to note that when such a coordinate grid is constructed, we are living on a (approximate) sphere in Euclidean space, and yet we also can consider the Earth as it exists in spherical geometry. Indeed, in spherical geometry, "lines" of latitude are not lines at all—that is, they are not the shortest path between two points on a sphere. Parallels are curves—a more general object—that are neither straight nor parallel, since the elliptic parallel property holds on a sphere, i.e., the nonexistence of parallel lines.[14] True lines on the sphere are called great circles, which (if we again think of the sphere embedded in Euclidean space), are the intersections of the Earth and any plane through the Earth's center. Any great circle intersects another great circle at exactly two points. The Equator is the sole line of latitude that is also a great circle.

Lines of longitude (called meridians), on the other hand, are halves of great circles that pass through the Earth's two poles. The Prime Meridian is defined to be 0 degrees East/West, and it was chosen to pass through Greenwich, England, the location of the Royal Observatory, a particularly significant place for astronomy and navigation.[15] Longitude itself is defined by the horizontal angle from the plane that intersects the Prime Meridian, which extends

[13]Recall that a point in a plane and a normal vector uniquely determines a plane in $\mathbb{R}^3$, so the equatorial plane is unique.

[14]Greenberg, *Euclidean*, 73-74.

[15]Graham Dolan, "The Royal Observatory Greenwich: A Brief History," *The Royal Observatory Greenwich*, accessed May 12, 2024, http://www.royalobservatorygreenwich.org/articles.php?article=1.



In this diagram, latitude is the orange angle represented by σ, and longitude is the blue angle represented by θ. Image from https://commons.wikimedia.org/wiki/File:Spherical_Coordinates_%28Latitude,_Longitude%29.svg.

from 180 degrees East to 180 degrees West.

3 Celestial Navigation

Celestial navigation is simply navigation using celestial bodies—in practice, the sun during the day and the stars and moon during the night—and it requires some knowledge of the celestial sphere and other astronomical definitions. The celestial sphere is an imaginary sphere of infinite radius surrounding the Earth that can be centered at Earth's center. The celestial bodies are projected onto the inside surface of this sphere. While the mathematical details of a sphere of infinite radius, which involve projective geometry, are beyond the scope of this paper, the intuition behind the concept is that the stars are so distant that, for practical purposes, we say that they are at an infinite distance so that their physical dimensions and variation in distance from the Earth are negligible (e.g., the size of a star or how many light years away from the Earth it is has no effect on navigational calculations).

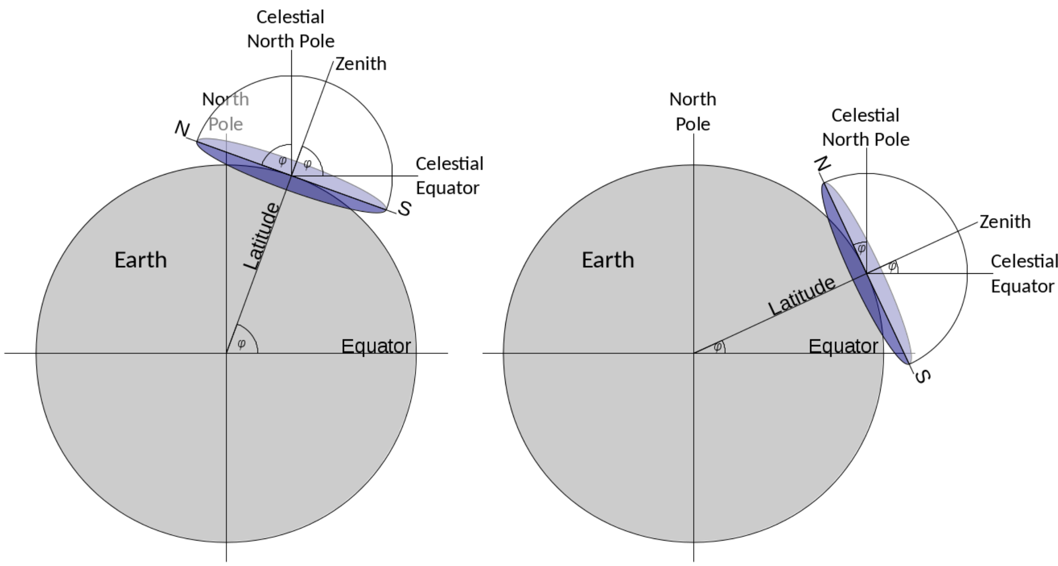


Diagram showing several features of celestial navigation such as the celestial equator. Here, the relationship between the altitude of the North Star, denoted by ϕ, and the latitude of observation, which is equal to ϕ, demonstrating that altitude of the North Star is sufficient for determining latitude. Image from Markus Nielblock, https://astroedu.iau.org/en/activities/navigating-with-the-kamal-northern-hemisphere/.

The celestial sphere is aligned with the Earth's poles, meaning that the Earth's poles are collinear with the celestial poles. The celestial equator is the intersection of the equatorial plane (of the Earth) and the celestial sphere at infinity. (Again, the details of intersection at infinity are beyond this paper's scope, but here is some more intuition: In a photograph of railroad tracks, the two tracks are parallel in actuality but in the photograph appear to coincide at the horizon. In projective geometry, we may consider these parallel lines to intersect at infinity.[16]) This relates to declination, which is the angular distance north or south of a celestial body from the celestial equator, a measure that relates particularly to the angle of the sun above or below the equator at any given point in the solar year. Declination is differentiated from altitude, which is the angular distance of a celestial body above the horizon at the place of observation. Declination of a celestial body does not change relative to an observer; altitude, on the other hand, does depend on one's latitude. As we will see shortly, measuring altitude is important for determining one's latitude.

When using the sun as a reference for navigation, the navigator must also consider the axial tilt of the Earth, which is the angle from the line normal to the plane in which the Earth

[16]Greenberg, *Euclidean*, 81.

orbits the sun. This axial tilt, whose measure is approximately 23.5 degrees, is responsible for the Earth's season, and causes the sun's declination to vary between 23.5 degrees North and 23.5 degrees South throughout the solar year. At the summer solstice, when the Northern Hemisphere is tilted toward the sun, the solar declination is 23.5 degrees North; this is when the altitude of the Sun at the Tropic of Cancer (located 23.5 degrees latitude North of the Equator) is 90 degrees, i.e., when the Sun is directly overhead the observer. At the winter solstice, the solar declination is 23.5 degrees South, and the altitude of the Sun at the Tropic of Capricorn is also 90 degrees. At the autumnal and vernal equinoxes, the solar declination is 0 degrees, meaning that the Sun is directly overhead at the Equator.

4 Determining Latitude & Longitude

A primary task of any navigator at sea is to determine one's latitude and longitude. Practically speaking, the problem of determining latitude is far simpler than determining longitude, and mariners have been able to easily estimate latitude for many centuries before longitude could be easily determined. From around the tenth century, Arab sailors used a tool comprised of a wooden board and knotted string called a kamal for finding relative latitude. Upon setting sail from a location, these sailors would hold the kamal in from of their face so that the bottom of the board touches the horizon and the top of the board touches Polaris, the North Star (or, in the Southern Hemisphere, the Southern Cross). When this was achieved, they would then tie a knot at the point of the string however far in front of them they had to hold the kamal. In their voyage, when they wished to find a line of latitude equal to that of their point of departure, they would sail North or South until the board of the kamal met the horizon and Polaris when they held the kamal at the length of the knot.[17] Such a tool does not provide the degree of latitude but a rough estimate of one's latitude relative to a known latitude.

Following similar geometric principles, one may also determine using the altitude of the Sun, given the solar declination of the day at which the observation is made. In other words, day or night, sailors could estimate latitude with the most rudimentary tools and simple

[17] Donald Launer, *Navigation Through the Ages* (New York: Sheridan House, Inc., 2009), 14-15.

Simple kamal.
Image from https://en.wikipedia.org/wiki/File:Simple_Wooden_Kamal_(Navigation).jpg.

geometric arguments. Instruments like the kamal became more sophisticated. For example, originating in the fourteenth century, a Jacob's staff (also called a cross-staff) is a staff with a vertical beam of fixed height and a ruled horizontal beam that may be shortened or lengthened. The navigator would use it in the same manner as the kamal, except the rulings allowed one to numerically estimate the altitude of a celestial body.[18] Eventually, instruments like this developed into the modern sextant, invented in the eighteenth century,[19] which is still sometimes advantageous over other modern navigational tools in some situations.[20]

Determining longitude is mathematically simple but was historically far more difficult than determining latitude. The simple mathematical principle is this: Since the Earth rotates a full revolution every 24 hours, it rotates 15 degrees longitude each hour. The navigator only needs to compare the time of observation of an astronomical event (such as the sun reaching its highest altitude at noon) at an unknown longitude with the time of observation of that same event at a known longitude. For example, if a sailor knows that it is noon at twelve o'clock in California and sails West in the Pacific until he observes that noon occurs at two o'clock according to a watch set to California time, then the sailor knows that he is two hours or thirty degrees West from California. Although this is simple in principle, prior

[18]Silke Ackermann, "Cross-staff," *Epact*, accessed May 12, 2024, https://www.mhs.ox.ac.uk/epact/article.php?ArticleID=5.
[19]Peter Ifland, "The History of the Sextant," accessed May 12, 2024, https://www.mat.uc.pt/ helios/Mestre/Novemb00/H61iflan.htm.
[20]Bowditch, *Navigator*, 220.

to the eighteenth century, a feasible means of accurate timekeeping, especially on a ship, was not yet discovered.[21] Instead, mariners had to rely on extremely tedious calculations of lunar distances or on a navigational method known as dead reckoning. Dead reckoning "determines a predicted position by advancing a known position for courses and distances."[22] In other words, using only one's speed and direction of travel and a known starting point, a dead reckoning estimates the mariner's new position. Navigators would often draw dead reckoning plots on nautical charts to keep track of position, usually taking a dead reckoning every half hour. They often used hourglasses to keep time and recorded their course (roughly synonymous with compass direction) on a tool known as a traverse board.[23] These plots often would be made on charts that used a Mercator projection,[24] since Mercator projections show lines of constant bearing (or direction), called rhumb lines,[25] as straight lines. However, these dead reckoning plots were often highly inaccurate, since a vessel is subject to ocean and wind currents that take it off course, and timekeeping was not precise.

A practical solution to the problem of accurately finding longitude did not come until the middle of the eighteenth century. The British Parliament passed the Longitude Act of 1714 which established a twenty-thousand-pound reward for anyone who could devise a practical method of finding longitude at sea accurate within half a degree.[26] The prize amount (which is almost $4 million in today's money) was significant, indicating that accurate navigation was a highly desired resource to the British government. The problem was solved with the invention of a marine chronometer by Englishman John Harrison several decades later, which allowed for accurate timekeeping at sea.[27] Eventually, the Prime Meridian was chosen to go through Greenwich, England, and thus navigators always had a reference longitude to

[21] Be Smart, "How We Solved the Greatest Riddle in Navigation," YouTube video, 14:15, November 8, 2022, https://www.youtube.com/watch?v=1o3$_{w}0Ypb$78.

[22] Bowditch, *Navigator*, 190.

[23] "Traverse Board," *Royal Museums Greenwich*, accessed May 12, 2024, https://www.rmg.co.uk/collections/objects/rmgc-object-43910.

[24] Bowditch, *Navigator*, 190.

[25] Eric W. Weisstein, "Mercator Projection," *MathWorld–A Wolfram Web Resource*, accessed May 12, 2024, https://mathworld.wolfram.com/MercatorProjection.html.

[26] Great British Parliament, "An Act for Providing a Publick Reward for such Person or Persons as shall Discover the Longitude at Sea" (1714).

[27] "Longitude Found: The Story of Harrison's Clocks," *Royal Museums Greenwich*, accessed May 12, 2024, https://www.rmg.co.uk/stories/topics/harrisons-clocks-longitude-problem.

compare to.[28] At this point, sailors could now know their latitude and longitude with only the stars, a tool to measure altitude, and an accurate chronometer.

5 Final Thoughts

Of course, this is only an abridged overview of early navigational techniques and how they arrived at the answers to the two basic problems of determining latitude and longitude. Furthermore, navigation has come a long way since the eighteenth century, with radar satellites providing the basis for much of our modern navigational infrastructure, and these modern techniques have much mathematical interest in their own right, but that are beyond the scope of this paper. In sum, mathematics has been an essential aide for the navigator, without which one would be hopelessly lost at sea, indicating that the two fields of mathematics and navigation are closely intertwined.

As a brief reflection, I was pleased with my presentation's reception by the class. People seemed interested in the subject, and some, like myself, would have been interested in studying more modern navigational technologies like radar satellites, but given time constraints, such a topic would have to be treated in a subsequent project. In my paper, I have condensed and cut some material on the Mercator projection, since the topic is somewhat tangential to the subject of this paper (even though there is much fascinating mathematics such as non-Euclidean geometry within cartography and map projections). Both cartography and modern navigation are compelling topics for a future project, one in which even more mathematical connections can be drawn.

[28] Dolan, "The Royal Observatory Greenwich."

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