Dungeons and Dragons

Probabilities, and math in game design

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November 2020

1 General Introduction of DnD

Dungeons and Dragons(commonly abbreviated as D&D or DnD) is a fantasy tabletop role-playing game(RPG) originally designed by Gary Gygax and Dave Arneson. It is a structured yet open-ended role-playing game. It is normally played indoors with the participants seated around a tabletop.

When playing, each player controls a single character, which represents an individual in a fictional setting. When working together as a group, these player characters (PCs) are often described as a "party" of adventurers, with each member often having their own area of specialty. During the course of play, each player directs the actions of their character and their interactions with other characters in the game. This activity is performed through the verbal impersonation of the characters by the players, while employing a variety of social and other useful cognitive skills, such as logic, basic mathematics and imagination.

The results of the party's choices and the overall storyline for the game are determined by the Dungeon Master(DM) according to the rules of the game and the DM's interpretation of those rules.

The mathematical part lies in the rules. While DnD is a game that requires a lot of storytelling from the DM, there is a set of rules that puts everything in a bucket. The most important function of these rules, is to determine whether something the player did, can succeed or not. The players would make all sorts of decisions in the game and it would be unfair and not fun for the DM to decide the success. That is why there is a certain set of rules, just to determine whether something works or not in the game. The success rate is based on the traits of the character, the difficulty of that event, and luck.

In this report I will be focused on the combat part.

2 D20

Before entering the combat, I need to explain what a D20 is.

D20 is 1 20-sided die. Dungeons and Dragons game relies on rolls of a 20-sided die, a d20, to determine success or failure.

$$E(1D20) = 10.5 \tag{1}$$

$$P(Getting - any - number) = 1/20 = 0.05 \tag{2}$$

These are the expectations for the dice. In other times you may see D8, D10, D6, these are 8-sided die, 10-sided die and 6-sided die. We use this form to represent dices.

3 Stats and Modifier

Stats would give you a certain modifier in related actions. For example, if you possess a strength of 16, you can get a +3 with every strength-related check. Your roll would then be 1D20 + 3. If you only have 10 strength, it would be 1D20 + 0. If you have 8, it would be 1d20-1. The exact form is here:

Ability Scores and Modifiers

Score	Modifier	Score	Modifier
1	−5	16–17	+3
2–3	−4	18–19	+4
4–5	−3	20–21	+5
6–7	−2	22–23	+6
8–9	−1	24–25	+7
10–11	+0	26–27	+8
12–13	+1	28–29	+9
14–15	+2	30	+10

Figure 1: Stats Modifier

4 Combat in DnD

Combat in DnD is a very complicated system. It contains several steps.

First, we need to decide the initiative. This is determined by dexterity rolls and whoever gets the highest number goes first. The dexterity roll would be 1D20 + dexterity bonus.

Next, someone would attack. First, to attack, the attacker needs to roll a hit rate. This is to determine whether the attack hits or not. This roll needs to compare to the enemy's armor class, AC. The AC is calculated with many different factors and it's a fixed value. If the hit roll is greater or equal to the AC, then the attack hits. Otherwise, the attack misses.

Thirdly, if the attack hits, you need to roll a damage roll. The damage roll will be related to the weapon and your stats. This will determine how much damage you can cause.

Then the attack is done, and the other person would take his or her turn. He or she would follow the same attack process.

Hit Points(HP) are the damage each person can take. If the HP goes to 0, the PC(player) would fall to the ground. The fight would go on until one falls to the ground, or runs away.

5 Specific Example

Let's take an example. Suppose we are simulating a fight between one player, and a common enemy. Let's first set the rank to 1, making it easy enough. Then we pick close-ranged fighter as the class since it requires no consideration of distance. Then we can take one single goblin as an example. The goblin is a common enemy and not too difficult for a single player. Most importantly, Wizards, the company that is in charge of the rules, has specifically stated the data of a goblin as a reference. This not only helped the DMs, but also helped me a lot in this calculation. Now let's check the stats of these two.

5.1 the Player

The player is a mountain dwarf fighter. This is a race that gets you 2 extra strength and constitution, which are both very useful for a fighter. Buying the stats in a proper way, can get both strength and constitution 16. This is a solid distribution. We gave dexterity 14 as well since it determines the initiative, order of combat. The rest of the stats can just be arranged anyway. The hit points, HP, is determined by the class alone, which in this case, is 12.

Class	Fighter(Level1)	
Race	Mountain Dwarf	
Hit Points(HP)	12	
Armor Class(AC	15(or 17)	
Strength	16	(+3)
Dexterity	14	(+2)
Constitution	16	(+3)
Intelligence	8	(-1)
Wisdom	12	(+1)
Charisma	10	(+0)

Figure 2: Player's Stats

Notice that the AC is flexible. First, there are two types of armors for a starting warrior. I picked a chain mail, which would provide the pc with an AC of 16. However, whether or not the pc holds a shield depends on the fighting style. If so, the AC would plus 2 and become 17. If not, it would just stay 15.

5.2 the Goblin

On the other hand, the goblin's stats are already prepared by Wizards, so there is nothing we need to do.

Race	Goblin
Hit Points(HP)	7
Armor Class(AC)	15
Hit Modifier	+4
Damage	1d6+2

Figure 3: Goblin's Stats

Notice that most of the cases, we only need the combat part of a goblin, so there is no need to figure out its specific stats. The hit modifier is a much easier expression in this case, showing us the number we should use when rolling the attack of a goblin. Simply adding this number to the D20, we would be able to tell whether the goblin's attack hit or not.

The goblin's dexterity is 14 by the way. This would be used later to decide the initiative.

6 Initiative

This part was originally not included in my speech. However, one of the classmates pointed it out so I decided to include it in my paper.

As introduced before, initiative, is the process of deciding who would go first. There are of course special cases like if you surprised your opponent or something. But in the common cases, this is determined by dexterity rolls and whoever gets the highest number goes first.

The dexterity roll would be 1d20 + Dexterity Modifier. Whoever possess a higher modifier gains more advantage in this process.

Let $a = MyModifier$, $b = Enemy'sModifier$

if $a - b >= 0$**:**

$$P(winning) = 1 - \frac{(20 - a + b)^2}{800} \tag{3}$$

if $a - b < 0$**:**

$$P(winning) = \frac{(20 + a - b)^2}{800} - \frac{20 + a - b}{400} \tag{4}$$

This is how the initiative would be calculated.

7 What happens in One Hit

In this section, we first try to figure out what would happen in one single hit from the player to the goblin. To simply the calculation, I mainly used expectations as comparisons.

7.1 Hit Rate

First let's calculate the hit rate. As introduced before, the hit would succeed if the hit roll is greater or equal to the AC of the enemy. The hit roll is consisted of 1d20 + hit modifier. Basically, you would roll a D20, then add your modifiers to it. The resulting number would be your hit roll. Then you can compare it to the enemy's armor class.

However, there are two special cases. If you roll a 1 out of 1d20, you would certainly fail the hit, called natural failure. If you roll a 20 out of 1d20, you would definitely succeed, and that would be a critical hit.

That is why we need three cases for the hit rate calculation.

Case1: When AC <= Modifier

$$P(Hitrate) = 0.95 \tag{5}$$

The only chance of failure is the natural failure.

Case2: When AC > 20 + Modifier

$$P(Hitrate) = 0.05 \tag{6}$$

The only chance of success is the critical hit.

Case3: When Modifier < AC <= 20 + Modifier

$$P(Hitrate) = \frac{21 + Modifier - AC}{20} \tag{7}$$

This time, only when the hit roll is greater or equal to enemy's AC, can the hit hits. So this is how you would calculate the hit rate in this case.

7.2 Weapon Damage

Now let's talk about different weapons. Here is a chart of weapon damages, just to give you an idea.

Martial Melee Weapons				
Battleaxe	10 gp	1d8 slashing	4 lb.	Versatile (1d10)
Flail	10 gp	1d8 bludgeoning	2 lb.	—
Glaive	20 gp	1d10 slashing	6 lb.	Heavy, reach, two-handed
Greataxe	30 gp	1d12 slashing	7 lb.	Heavy, two-handed
Greatsword	50 gp	2d6 slashing	6 lb.	Heavy, two-handed
Halberd	20 gp	1d10 slashing	6 lb.	Heavy, reach, two-handed
Lance	10 gp	1d12 piercing	6 lb.	Reach, special
Longsword	15 gp	1d8 slashing	3 lb.	Versatile (1d10)
Maul	10 gp	2d6 bludgeoning	10 lb.	Heavy, two-handed
Morningstar	15 gp	1d8 piercing	4 lb.	—
Pike	5 gp	1d10 piercing	18 lb.	Heavy, reach, two-handed
Rapier	25 gp	1d8 piercing	2 lb.	Finesse
Scimitar	25 gp	1d6 slashing	3 lb.	Finesse, light
Shortsword	10 gp	1d6 piercing	2 lb.	Finesse, light
Trident	5 gp	1d6 piercing	4 lb.	Thrown (range 20/60), versatile (1d8)
War pick	5 gp	1d8 piercing	2 lb.	—
Warhammer	15 gp	1d8 bludgeoning	2 lb.	Versatile (1d10)
Whip	2 gp	1d4 slashing	3 lb.	Finesse, reach

Figure 4: Weapon Damage

Now let's calculate the weapon damage. The damage of weapons are all recorded in such manner, 1dX. This means, we should roll an X-faced dice, then plus the modifier. Then we can get the damage. The expectation of a hit goes as follows:

$$E(one-hit) = \frac{X+1}{2} + modifier \tag{8}$$

However, this is assuming that the blow actually hits. If we take all situations into consideration, the damage expectation would change.

$$E(one-hit) = (P(Hitrate) - \frac{1}{20}) * (\frac{1*(X+1)}{2} + M) + \frac{1}{20} * (1*(X+1) + M) \tag{9}$$

You may notice, that an extra case appeared in this formula. That is the critical hit mechanism. Basically, if you roll a 20 out of D20, you get a critical hit. You would roll 2 damage dices instead of 1, resulting in a higher expectation of damage in this case.

7.3 Two Fighting Styles

There are two fighting styles. You can use a two-handed weapon. Or you can use a one-handed weapon, and hold a shield in the other hand, giving you 2 more extra armor class. Now let's analyse the weapon damage of each one of them.

Sword and Shield First, let's check the damage expectations of single-handed weapons. Let's take our warrior as an example. She has a +5 modifier in her attack roll, and a +3 modifier in her damage roll. Now, the only variables are the weapon and the enemy's AC. I made a chart for three typical single-handed weapons.

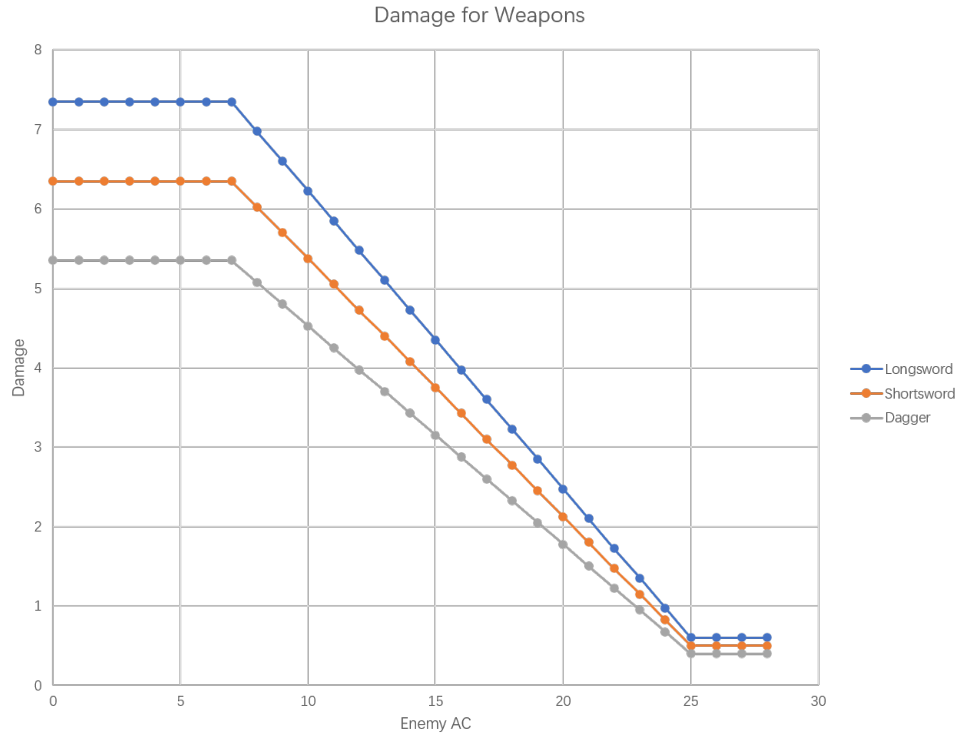

Figure 5: Single-Handed Weapon Damage

From this graph, we can see that longsword does more damage than the other two. It is interesting how the weapon matters less when the enemy AC gets higher. The weapon damage only matters when the hit rate is not too low. If the hit rate is too low, which weapon you use would hardly matter.

Two-Handed Weapon Now let's check two-handed weapons.

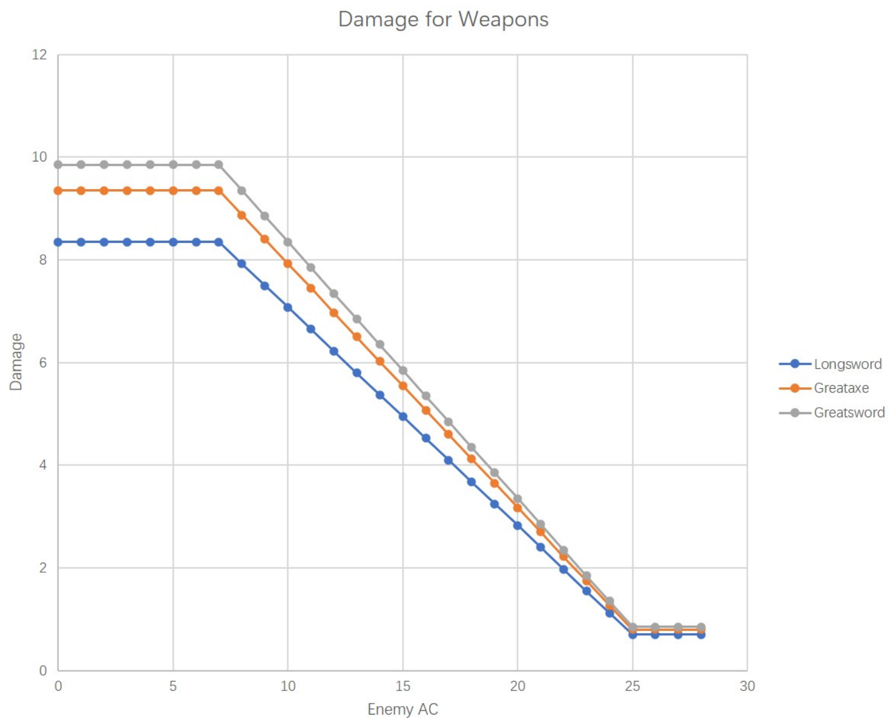

Figure 6: Double-Handed Weapon Damage

We can see that Greatsword, in this case, possess the largest damage expectation. We can see, in this case, the damage expectation of two-handed weapons is much larger than that of one-handed weapons. It is, of course, very reasonable. But, does that mean two-handed is a better way of combat than sword and shield? We will analyse that problem in the next section.

8 the Duel with Goblin

In this section, we will finally start analysing the actual 1on1 fight between our main character and the goblin. Our goal is to use probabilities to figure out the best strategy against a goblin, for our PC.

8.1 Initiative

The initiative is quite simple. Since both our PC and the Goblin possess dexterity 14, the probability of getting initiative is the same for both of them.

$$P(initiative) = 0.5 \tag{10}$$

8.2 Fighting Expectations

In this part, I will use expectations on each step and get the expected result for each fighting style. In the end, the fighting style that makes smallest loss would be the best.

Sword and Shield In this case, we will give our fighter a longsword. This sword possess a damage of 1d8. Our fighter, as we said before, has a hit modifier of 5 and an damage modifier of 3. The goblin, on the other hand, has a hit modifier of 4 and a damage modifier of 2.

That gives us the following data:

$$P(Hitrate) = 0.55 \tag{11}$$

$$E(PCdamage) = 4.35 \tag{12}$$

$$P(gettinghit) = 0.4 \tag{13}$$

$$E(gobdamage) = 2.5 \tag{14}$$

If the player got the initiative:

First blow, goblin takes 4.35. Next turn, player takes 2.5. Third turn, goblin takes 4.35 again. Since $4.35 + 4.35 > 7$, which is the goblin's hit point, it would die after this turn. The player took 2.5 damage overall.

If the player didn't get the initiative:

First blow, player takes 2.5. Next turn, goblin takes 4.35. Third turn, player takes 2.5. Forth turn, goblin takes 4.35 again. Since $4.35+4.35 > 7$, which is the goblin's hit point, it would die after this turn. The player took 5 damage overall.

There is a 0.5 chance of taking 2.5, a 0.5 chance of taking 5. So the overall expectation of damage is 3.75, if the player wants to take down a goblin in this manner.

Two-Handed Now let's go over two-handed case. This time AC would be 15, and the player would use a greatsword.

That gives us the following data:

$$P(Hitrate) = 0.55 \tag{15}$$
$$E(PCdamage) = 5.85 \tag{16}$$
$$P(gettinghit) = 0.5 \tag{17}$$
$$E(gobdamage) = 2.925 \tag{18}$$

If the player got the initiative:

First blow, goblin takes 5.85. Next turn, player takes 2.925. Third turn, goblin takes 5.85 again. Since $5.85 + 5.85 > 7$, which is the goblin's hit point, it would die after this turn. The player took 2.925 damage overall.

If the player didn't get the initiative:

First blow would be taken by the player. A similar process would happen. The goblin would die and the player would take a damage of 5.85.

So the overall expectation of hurt is 4.3875. This is larger than 3.75, so I suppose that the former one is the better fighting style in this case.

8.3 Rate of Failure

All these analysis brought us to the final question. What is the rate of failure for our player to defeat the goblin?

My idea is to calculate the rate of each failed case and add them together. However, it didn't work out so well. The equation turns into something like this:

P = 0.5(didn't kill the goblin in the first turn * goblin killed me in 1 turn + didn't kill him in two turns*killed me in 2 turns + 3 hits didn't kill * killed in 3 turns + ...)+ 0.5(goblin killed me in first turn + didn't kill goblin in 1 turn*goblin killed me in 2 turns + ...)

In this formula, the former half is the case where I got the initiative, the latter half is the case where the goblin got the initiative.

I failed to solve this formula and I am still working on it. This is a project that still requires working on.

9 Conclusion

Even though my final analysis of the failure rate failed, I feel like the former analysis of expectation would still work in this case. Since each hit is independent, adding the expectations together would probably be fine. So in this case, holding a shield is probably wiser than holding a greatsword.

This is not just for getting a convenience in the game. These calculations are also important for game design I believe. If the shield and sword is a better choice in this case, what would be a better case for the greatsword? You can see that there are other characters next to the weapons showing their different traits. The two weapons I picked are simply the ones with highest damage expectations in the exact situation I designed. The traits of other weapons would give them some unique functions in other cases.

In the end, can we use these theories and calculations in our real-life managements? I believe we can. It would be much different and more complicated, but there is still some ways to analyse them. I do hope that with these tools, we can give everyone in the society the suitable 'sword' they need.

10 Reference

Wikipedia: Dungeons Dragons

DnD, 5th edition, player's handbook