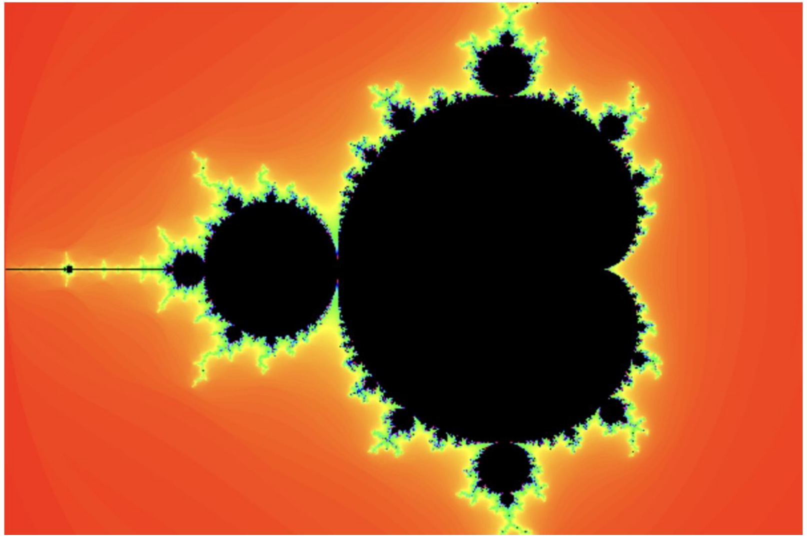


Mandelbrot Set: The set of numbers in the complex plane for which the orbit with initial condition z=0 remains bounded, with the function $z_{n+1}=z_n^2+c$.

Fractals and Cities

"Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line." – Mandelbrot

Grace Lupold

Introduction

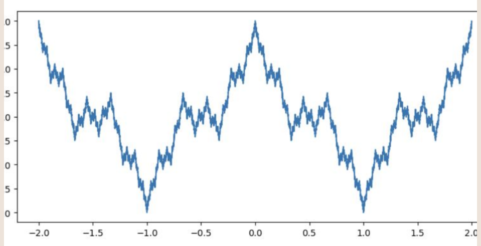
Fractals: A way to describe irregular patterns. For a shape to be fractal it must have:

1. Self-similarity: No matter how much you magnify, the shape is the same
2. Irregularity: Fractal dimension

There are many different types of fractals, some being purely mathematical, but fractal-like patterns appear in nature, cities, artwork, and more.

A Brief History

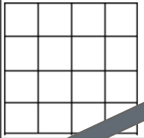
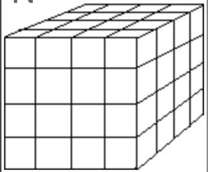
$$f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x)$$



- The mathematician Benoit Mandelbrot coined the term in the 1960s with his discovery of the Mandelbrot set
- However, fractals had been around for years prior
    - Gottfried Leibniz was the first, where in the 17th century he used the term ‘fractal exponents’ to explain the scaling properties of recursive self-similarity
    - They were then further explored through Karl Weierstrass’ discovery of a function that is continuous everywhere but not differentiable, which spurred Cantor’s discovery of Cantor Sets (self similarly lines and fractal patterns)
    - Koch created the Koch snowflake in 1904
    - Sierpinski Triangle was created in 1915

Calculating Fractal Dimensions

- Look at N (the number of self-similar copies) and S (the scaling factor).
- In our case, the equation relating N, S, and D (the dimension) is $N = S^D$

Explanation	**Image**	**Exponent gives the dimension**
Notice that a line segment is *self-similar*. It can be separated into $4 = 4^1$ "miniature" pieces. Each is 1/4 the size of the original. Each looks exactly like the original figure when magnified by a factor of 4 (magnification or scaling factor).		$4 = 4^1$ pieces
The square can be separated in to miniature squares. If the smaller square is magnified (*scaled*)4 times then it is identical to the larger square. However, we need $16 = 4^2$ pieces to make up the original square figure.		$16 = 4^2$ pieces
The cube can be separated into $64 = 4^3$ pieces. Again, these pieces need to be enlarged (*scaled*) by a factor to 4 to generate the larger square.		$64 = 4^3$ pieces

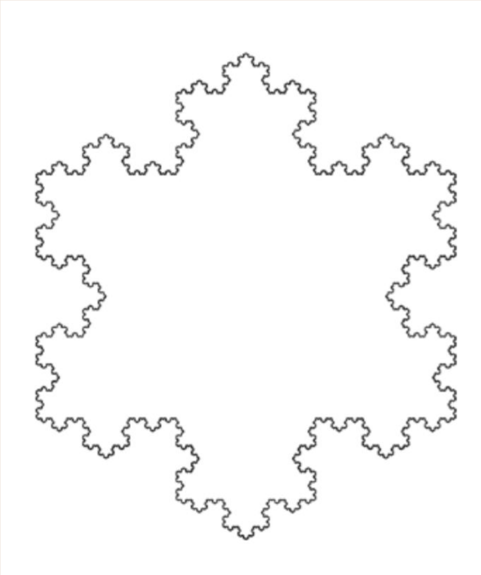
D

N

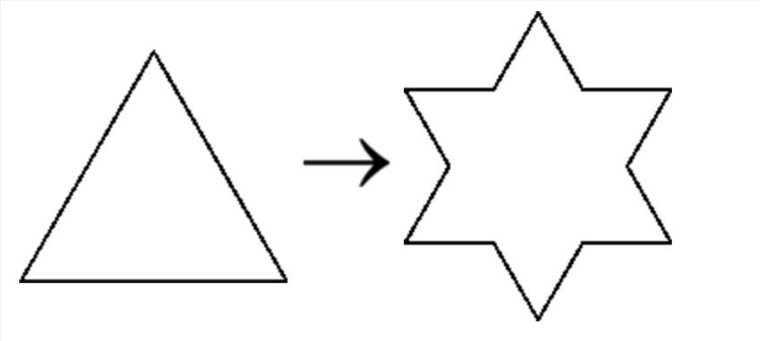
S

Solving $N = S^D$ for D by taking logarithms, we get $D = \log N/\log S$.

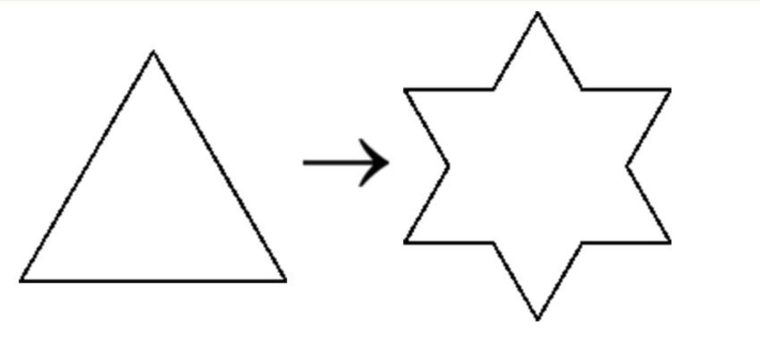
Example: The Koch Snowflake



- This is an example of an iterated function system fractal – fractals based on simple plane transformations
- It is made by taking an equilateral triangle, taking out the middle third of each side, and adding two more line segments to make another triangle



The Koch Snowflake Dimension



D = logN/logS.

What are N and S in this case? Remember N is the number of self similar copies and S is the scaling factor.

Since 1 line becomes 4, N = 4, and each line becomes ⅓ of the original size, so S is 3. Thus, the dimension is log 4/log 3 = 1.2619

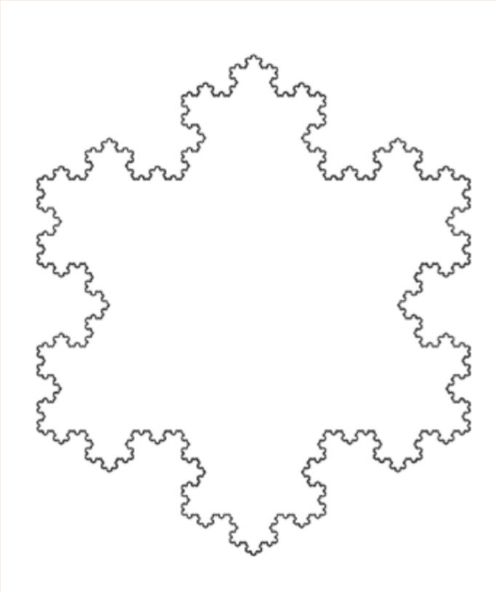
This is just one way to calculate the dimensions of a fractal, this way was known as the self-similar method but there are 2 other methods known as the Richardson Method for calculating a dimensional slope, and the box-counting method for determining the ratios of a fractals area over volume (often used in nature)

So How Long is the Perimeter?

Infinitely long!

At each iteration, every line is replaced by 4 lines of 1/3 the length (a factor of 4/3).
So, we can solve the $\lim_{n \to \inf} (4/3)^n$

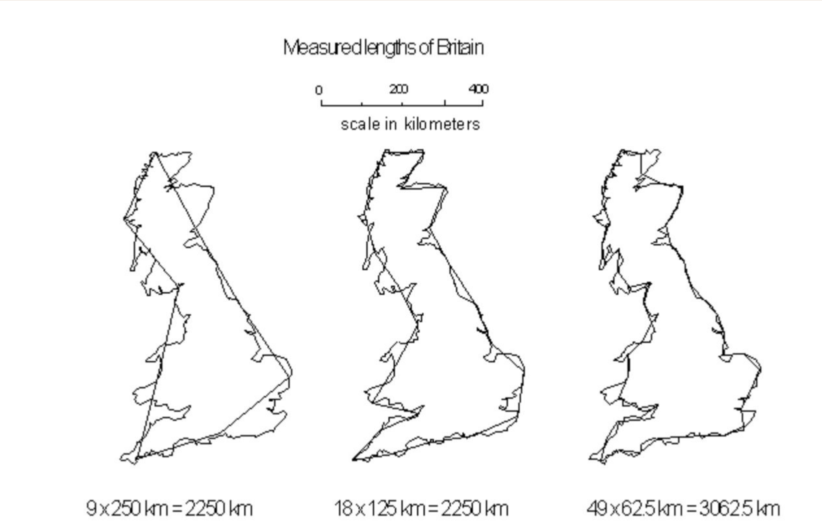
Because (4/3) > 1, the length increases with n, and thus the Koch snowflake has an infinitely long perimeter.

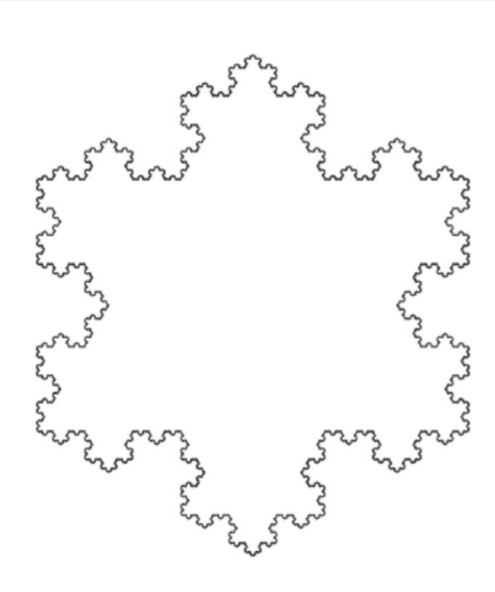


What about the area?

Not infinitely long!

This website explains how the area is finite, where if each side has length 1 it is 2*sqrt(3)/5.

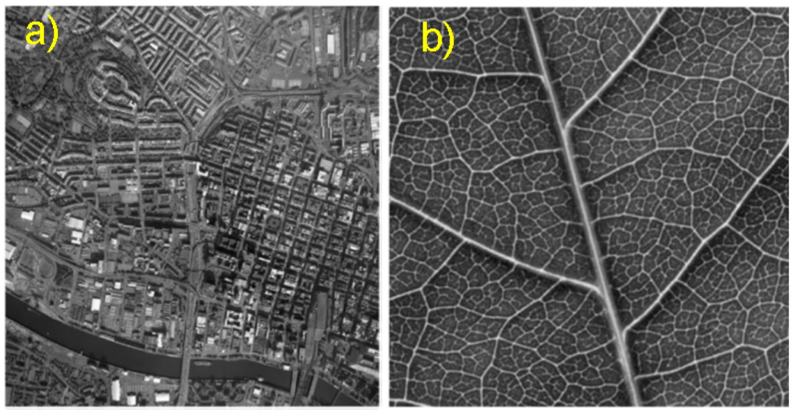




This happens with coasts too! Coasts with higher fractal dimensions (more complicated and jagged edges) will increase in perimeter length the more specific you are.

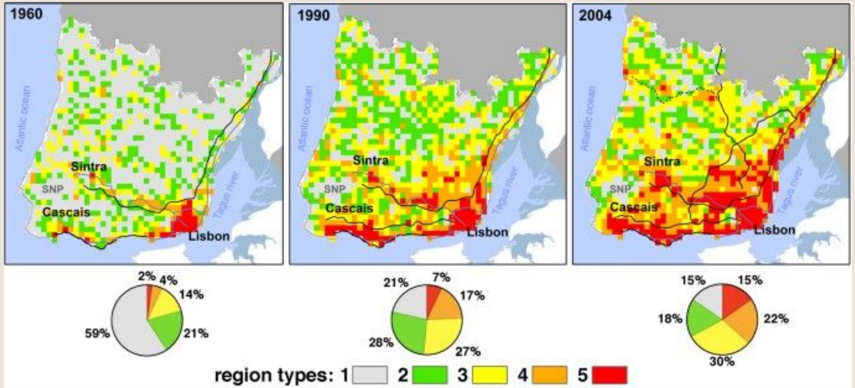
Connections to Urban Theory and Development

- Urban areas can be considered to be complex dynamic systems (with a spatial pattern determined by land use dynamics)
- Fractals can be used to describe the complexity of the urban spatial form
- They can also be used to show growth and maximize the amount of nature people have access to



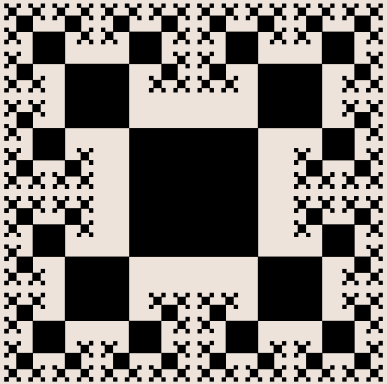
Using Fractal Dimension to Describe Cities

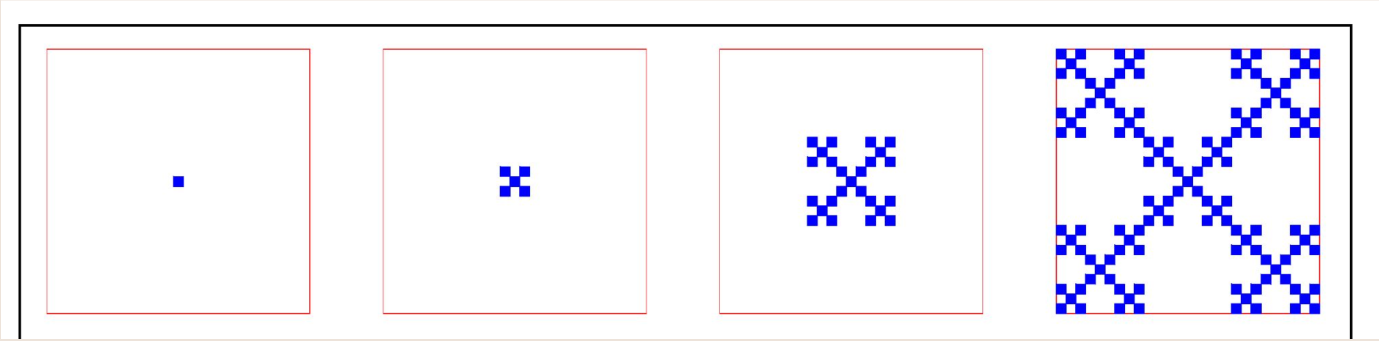
- Usually, density is used to describe the spatial distribution of a population and activities in a geographical space
  - Conceptually, this relies on a direct relationship between the population and the occupied surface
  - This Euclidean geometric approach fails to describe the irregularities and fragmentation of a city
- To calculate the dimensions, one can use the box method, where you place a grid over the city and counts the number of boxes that are filled
- Lower fractal dimensions = dispersed areas, fractal dimensions closer to 2 = compact, built-up areas
- Comparing cities worldwide, there was an average dimension of 1.7
- Growth patterns in cities also show an increase in the fractal dimension over time



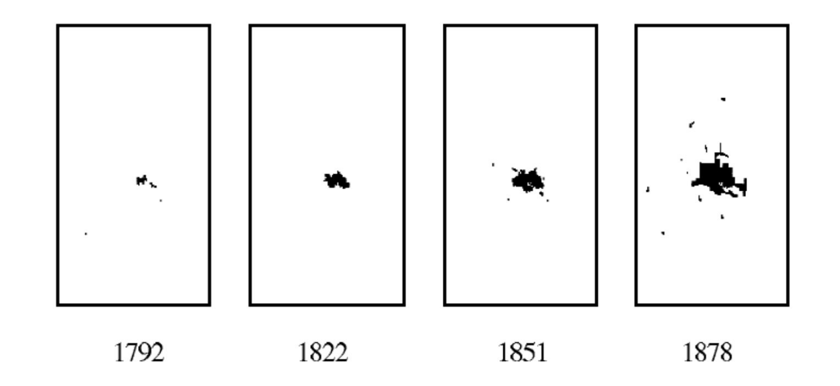
Fractals and Urban Growth

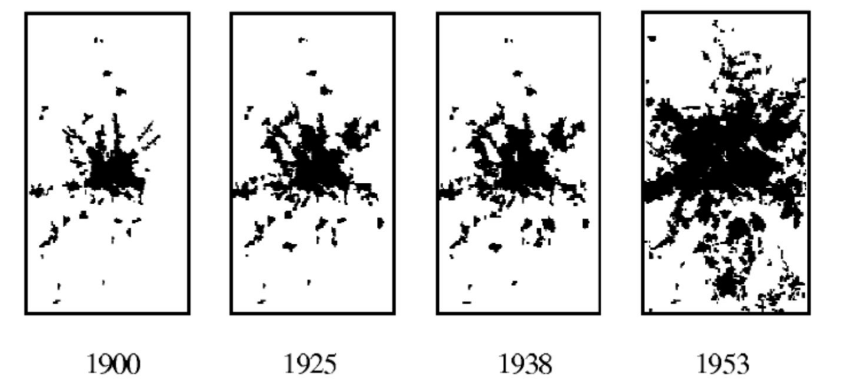
- The more a city spreads in surface, the more fragmented and ‘shredded’ its appearance become
- The borders of urban areas do not follow Euclidean geometry
  - One way this happens is that every person living in a suburban area wants to live closer to green areas
  - When implemented in fractal manners, the entire population can take advantage of natural areas without spreading too far out (sierpinksi carpet)
- Many towns and cities grow through slow, incremental (iterative) processes
- Think about the new construction going up, the expansion of the college





Seeing the Growth of a City: Baltimore, Maryland





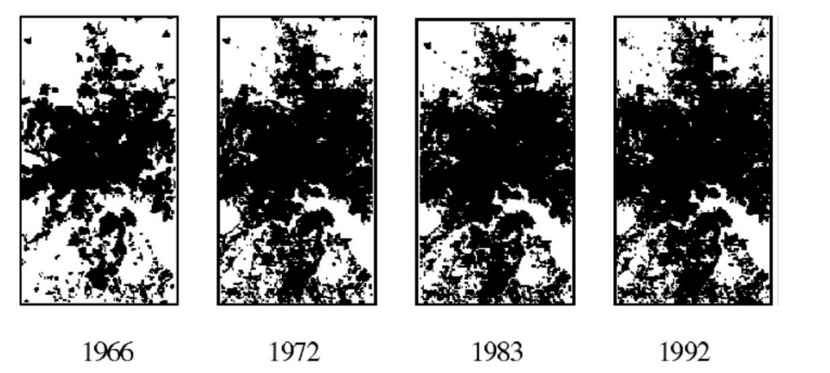
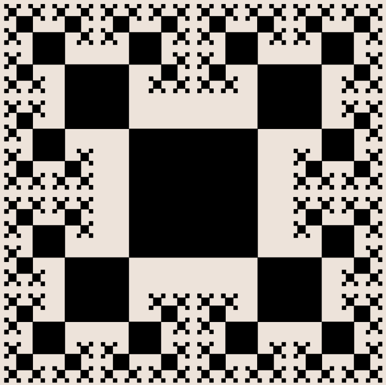


Figure 7.   Urbanized areas in Baltimore, MD in 12 selected years.



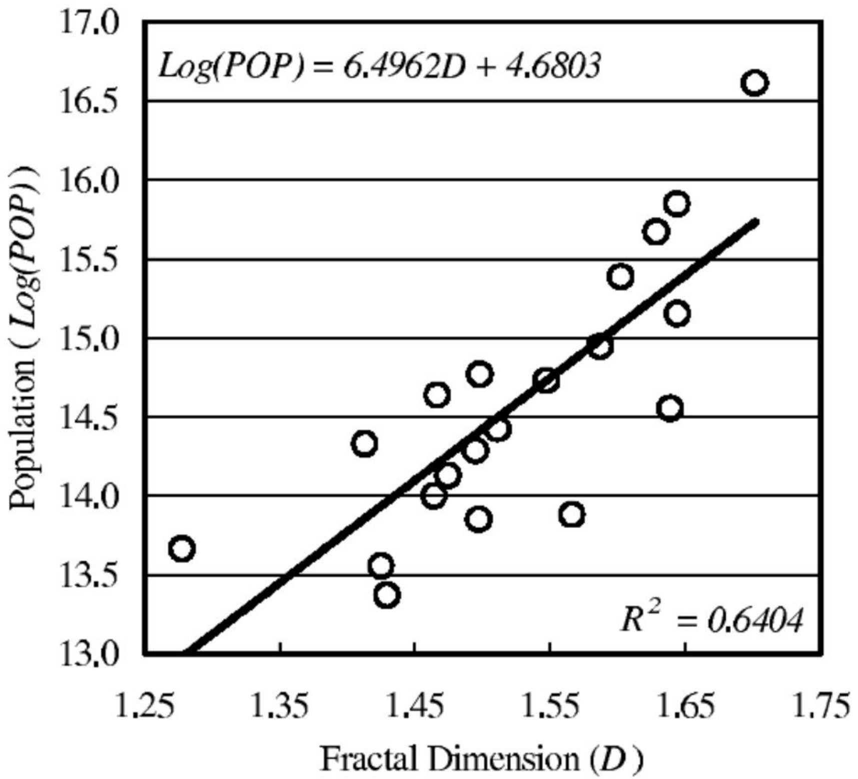
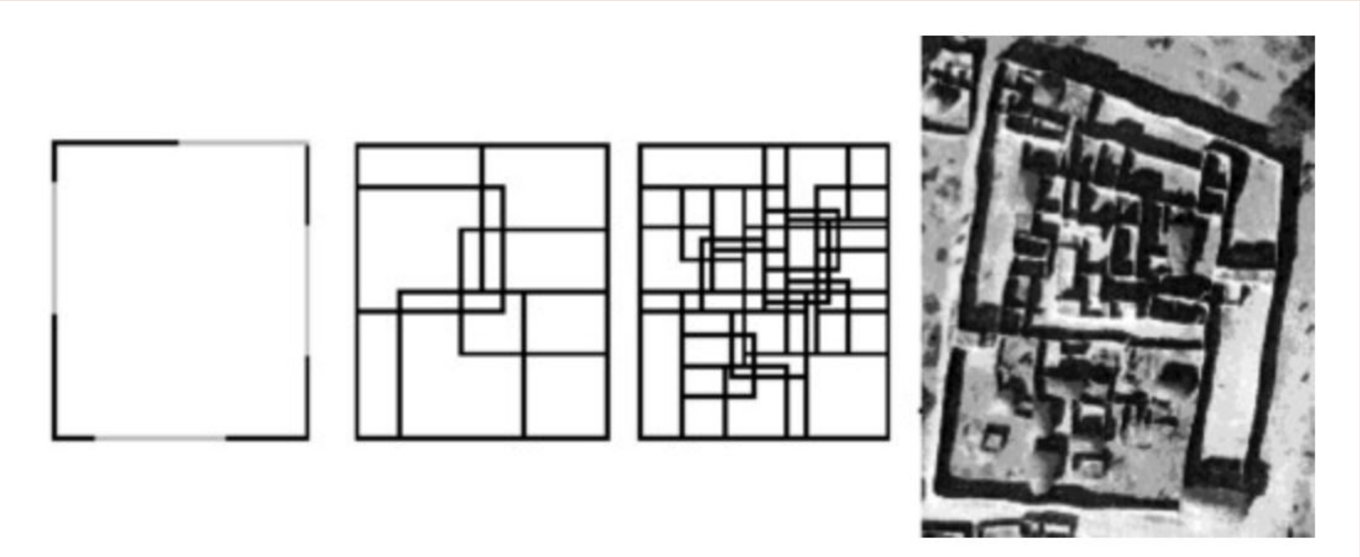


Figure 5. *Log* (*POP*) as a linear fraction of *D* for 20 US cities.

Display of how fractal dimension is related to population growth

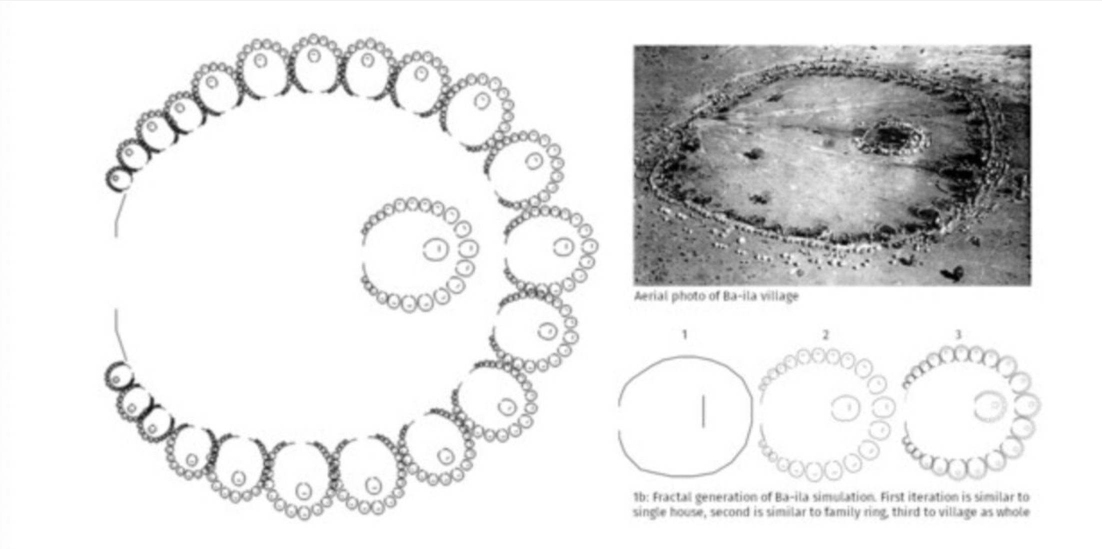
Ancient Fractal Cities



The Logone-Birni in Cameroon, founded by the Kotoko people

- Rectangular enclosures grew from the preexisting walls.
- “A man would like his sons to live next to him, and so we build by adding walls to the father’s house”
- Defensive advantages, and display of familial hierarchies

Ancient Fractal Cities



- Each circular enclosure has a family dwelling, with a livestock pen in the front and an altar in the back
- Another example of the visual hierarchy of family

The Ba-ila settlement of Southern Zambia

Why?

- Pre-modernist cities were built over time based on pedestrian models
    - Aka continuous incremental additions
    - Everyone living in a city had to have access to housing, food, often a church, a market, a place of work, etc. They needed a human-scaled city fit to their needs
- Large fractal cities tend to grow by absorbing smaller villages, creating a city that is a collection of smaller “cities” (villages), which are all a collection of neighborhoods
- They also were clear displays of hierarchies and faith relations

Why Don’t Our Cities Look Like That?

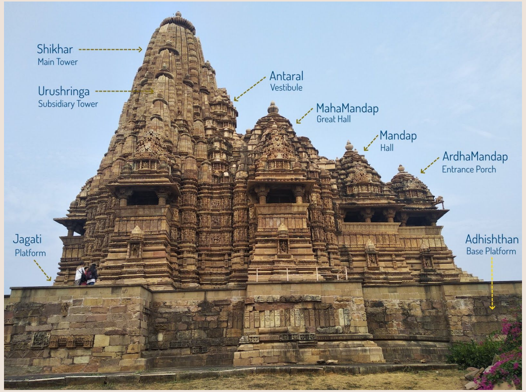
- **Cars**
- Our cities, based on a block structure, were not slowly and organically created by everyone, they were designed by a few urban planners that emphasized car over pedestrian movement
  - Car-based infrastructure removed the human-scale connections and growth (as well as physical space)
- Makes a fractal-based argument for a walkable city!
- Additionally, it explains the growth of suburbs (a want for a connection to nature) in the outskirts

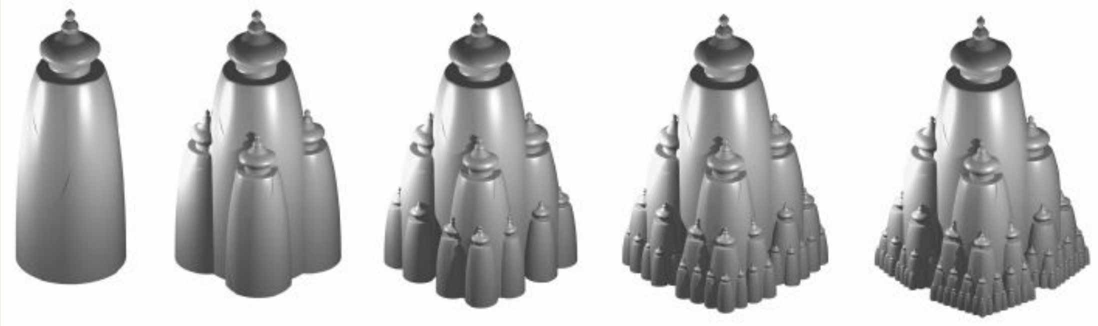
- Manhattan has 36% of its area dedicated to streets
- 96% of new yorkers walk to and from public transit

Fractals and Religious Architecture

European Cathedrals of the Gothic, Renaissance, and Baroque architecture

Fractals and Religious Architecture

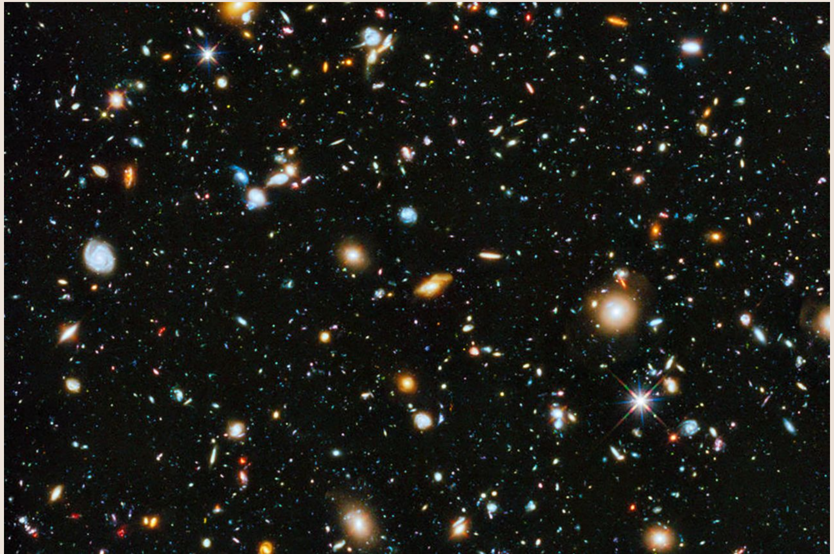


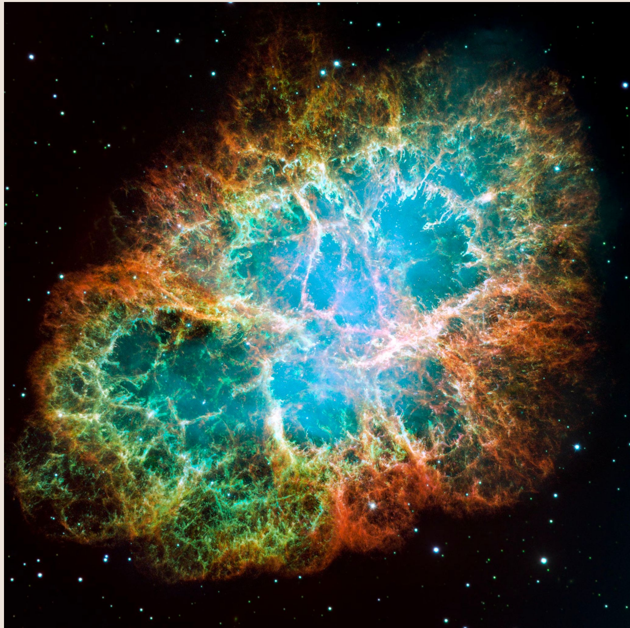


Hindu temples

Fractals and Faith

- Some argue that fractals are “a way to view eternity,” and thus proof of the existence of a creator
- “Only intelligence creates order from disorder.”
- “Fractals are the geometry of eternity”





Questions?

Resources

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