Math 400: Fractal Geometry and Its Connections to Urban Planning and Development

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Abstract

Fractals are geometric shapes characterized by self-similarity and a non-integer dimension. They are found everywhere, from stalks of broccoli to cantor sets, and the math behind them carries many important applications. By utilizing the relationship between the number of self-similar copies and the scaling factor of an object, non-integer dimensions can be found to describe the infinite complexity of fractal shapes. Extending the equation to a brute-force method, the box-counting technique allows the fractal dimension of cities to be calculated. Since urban areas have many of the same traits as fractals (irregularity, fragmented borders, nested formations, and more), calculating fractal dimensions of cities proves valuable in explaining urban growth, characterizing the irregularity of cities, and maximizing the connections of a population to nature. This paper seeks to analyze the importance of fractal geometry for geography and urban planning, demonstrating that the traditional Euclidean geometric techniques are not always the best way to describe the spatial patterns of cities.

1. Mathematical Connections

In choosing a topic for this presentation, I wanted it to be something beyond the scope of the required math curriculum. Seeing as William and Mary does not offer a class based on fractal geometry, I thought it would be interesting to teach both myself and the class a bit more about them. What especially piqued my interest was how the field traverses both theoretical and applied math. On the one hand, fractals are abstract as they connect to chaos theory and irregularity, but another side of the geometry shows how simple examples and formulas can analytically describe their patterns and behavior. One concept that highlights this well is the fractal dimension, which encapsulates the infinite complexity of fractals using a single numerical value and a simple equation. When I discovered how urban planning and development connect to these dimensions, I knew it would be a perfect fit. By connecting fractals to cities, I utilized theoretical and practical fractal geometry to explain historical civilizations, urban/population growth, and spatial dynamics. In making this connection, I wanted to show how we can benefit from using properties of fractal geometry to make predictions and calculations about our world.

2. Fractal Geometry

Fractal geometry is a branch of mathematics focused on the study of irregular shapes known as fractals. While Benoît Mandelbrot is the mathematician who often gets the credit for creating the field in the 1960s (with his discovery of the Mandelbrot set), many of the fundamental concepts of fractal geometry have roots dating back centuries. The earliest documented discovery is credited to Gottfried Leibniz in the 17th century, who introduced the term ‘fractal exponents’ to describe the scaling properties of recursively self-similar objects. [4]

By definition, fractal geometry studies the properties of fractal objects, which are shapes that have the traits of self-similarity and irregularity. Self-similarity is defined as the ability to magnify the shape to any scale and still have the same structure. For example, when looking at a stalk of broccoli, no matter how much one zooms in, it always appears to have a stem and a floret of buds. The second fractal property of irregularity describes the non-integer dimension. Unlike the traditional dimensions (1d, 2d, 3d, etc.), fractal dimensions are decimal numbers. This number represents how "close" the object is to an integer dimension, so a dimension close to 1 would be a shape closer to a line, and one close to 2 would be a shape that almost fills up the entire plane. [3] [11]

To calculate this dimension, mathematicians utilized the scaling factor, the number of self-similar copies, and integer dimensions for lines, planes, and 3d objects to find a relationship and extended it to fractals. For example, a line segment in the first dimension can be broken up into 4 sections (each of ¼ the size). So the dimension is 1, the scaling factor is 4, and the number of self-similar copies is also 4. The relationship between these traits can then be represented as $4 = 4^1$. In the second dimension, a square plane can be filled with 16 squares (which are self-similar copies) that are each ¼ of the size. In this case, the scaling factor is 4, the dimension is 2, and the number of self-similar copies is 16, which can be represented as $16 = 4^2$. Thus, representing N as the number of copies, S as the scaling factor, and D as the dimension, the equation used to define an object is $N = S^D$. Solving for D, the equation to find the fractal dimension of an object is $D = \frac{logN}{logS}$. [11]

For example, the Koch snowflake (Figure 1), which was created in 1904 by Helge von Koch to give a geometric definition of a similar function, is created by transforming each side of an equilateral triangle by taking out the middle third and adding in edges to create another triangle that is ⅓ of the original size (Figure 2). So every iteration transforms 1 line into 4 lines, each of ⅓ of the size, and its dimension can be expressed as $D = \frac{\log 4}{\log 3} = 1.2619$. In context, this means that the shape is closer to the dimension of a line than a plane. [4] [6]

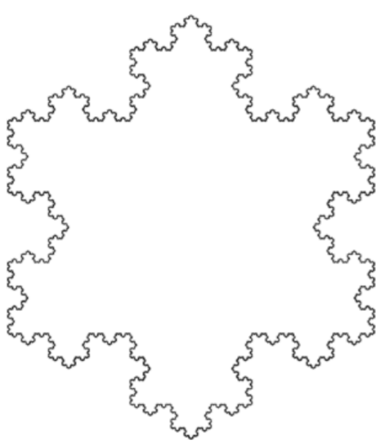

Figure 1. The Koch Snowflake [6]

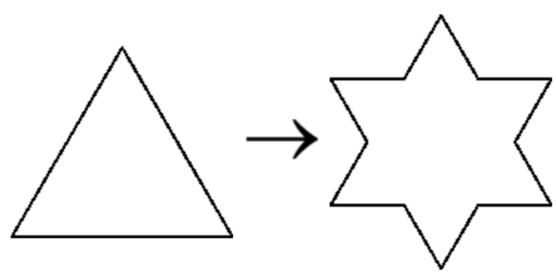

Figure 2. The first iteration of the Koch Snowflake [6]

However, what becomes more fascinating is that the perimeter of the snowflake is infinitely long, while the area is finite. Each iteration increases the total length of the shape by a factor of $\frac{4}{3}$, and thus for n iterations, the length will be $(\frac{4}{3})^n$, which approaches infinity as n grows. To find the equation of the area, one can use geometric series rules and the area of an equilateral triangle. Specifically, if the length of each side of the original triangle is s, the area of the triangle before the first iteration is $\frac{\sqrt{3}}{4}s^2$. For the next iteration, the side length s becomes s/3 and the equation is $\frac{\sqrt{3}}{4}s^2 + 3\frac{\sqrt{3}}{4}(\frac{s}{3})^2$. Continuing this logic, the area at the kth iteration is $\frac{\sqrt{3}}{4}s^2(\frac{3 \cdot 4^{k-1}}{9^k})$, which is a geometric series that can be simplified to $\frac{2\sqrt{3}}{5}s^2$. Thus, through fractal geometry, it is possible to have a shape that is infinite when looking at its perimeter, and finite when looking at its area. [6] [10]

This idea applies to a phenomenon known as the coastline paradox, where coasts with higher fractal dimensions (as in those that are more fragmented) will dramatically increase in perimeter length as they are measured more specifically. This can make it difficult to get an accurate or standard length of coastlines for naturally fractal boundaries. In his research, mathematician Lewis Fry Richardson discovered that coastline measurements can vary by upwards of 20%, depending on how specific the “measuring stick” is. [1]

Beyond noting the reasoning behind the coastline paradox, Richardson also pioneered a new way to calculate these dimensions, as the traditional way described above only works for perfectly mathematical fractal objects. This method, known as the Richardson Method, uses ‘rulers’ of varying lengths to measure an object's perimeter and then plots the results on a log-log graph, measuring the slope. While slightly less accurate, this method allows mathematicians to measure fractal properties of shapes that are not perfectly self-similar. [1]

The final way to calculate fractal dimensions is known as the box-counting method, which is used to estimate the dimension of objects when numerical formulas and the Richardson method fail. It entails placing a grid over the object and counting the number of squares that contain the shape. Then, one plugs it into the equation $D = \frac{\log n(a) - \log n(b)}{\log(1/s_a) - \log(1/s_b)}$, where n() is the number of squares containing the image and s is the scale of the grid. There are two measurements (for two different grid sizes) so that the average can be taken. While less accurate, this is the method that is used the most for calculating fractal dimensions of cities, clouds, leaves, and many other naturally occurring fractal patterns. In both the Richardson and box-counting methods, the purely mathematical way to calculate fractal dimensions (the first method discussed) is modified to accommodate the imperfections of natural fractals. [1]

3. Urban Planning and Development

Although fractal geometry gained widespread recognition only in the late 20th century, its presence in architectural design has been evident in indigenous cultures and villages for centuries. For example, the Logone-Birni in Cameroon (Figure 3) build their community with large, rectangular-shaped, dwellings where walls and rooms are slowly built up within the space of the original building. As children in the family grow up and become independent, they build walls inside the larger house to give them a smaller room. Another example is the Ba-ila settlement of Southern Zambia, which consists of a series of circular enclosures of growing size, where the small end is for the livestock and the largest building in the back is for the altar and head of the household. Additionally, each smaller circle of houses represents a family line enclosed within the larger circle of the village, where more prominent families have larger homes. In both instances, these architectural formations not only serve a practical purpose (usually for protection), but they are a visual manifestation of social hierarchies and the faith of the community. [12]

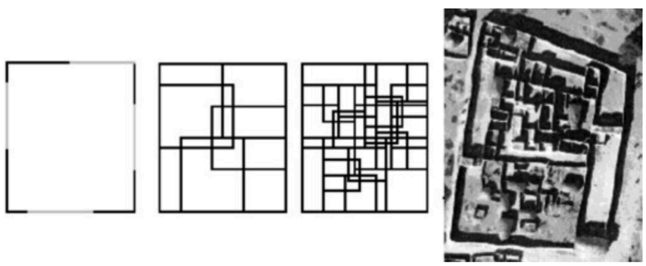

Figure 3. The Logone-Birni [12]

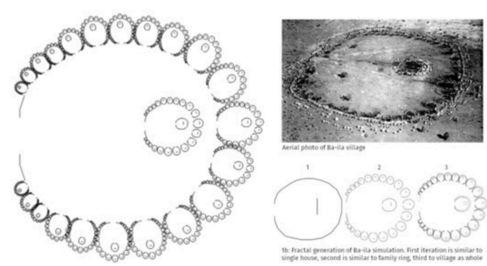

Figure 4. The Ba-ila [12]

Modern cities today are less noticeably fractal, as they are built prioritizing cars (which remove human-scaled connections) and often use grid systems and commercial/residential zones (removing self-similarity). Additionally, hierarchical structures and religious undertones are less present in today's urban landscapes as society is less faith-focused, and many people choose to commute. But in a general sense, all cities are fractal. Looking at urban spaces as nested self-similar communities, any city can be viewed in a fractal sense. For example, a city is composed of towns, which is composed of villages, which is composed of neighborhoods, and so on. Moreover, urban expansion and development frequently follow fractal patterns, evolving incrementally over time and adopting irregular borders to maximize access to nature. See Figures 5 and 6 to see how the growth of Baltimore, Maryland resembles a fractal. [2] [5] [8]

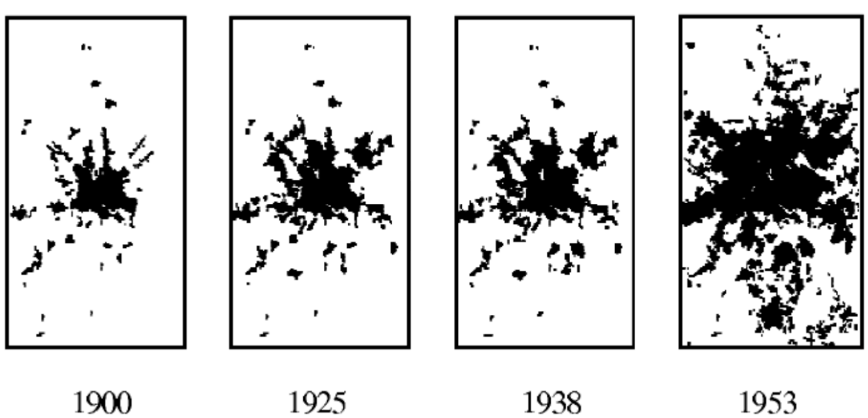

Figure 5. The growth of Baltimore, Maryland [8]

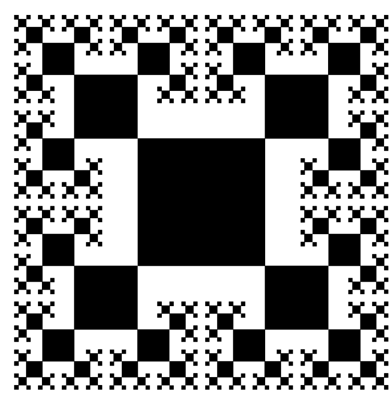

Figure 6. T-square Fractal

These irregular spatial patterns in cities cause traditional Euclidean geometry to often fall short in accurately characterizing an urban area. For example, looking only at the surface area or population density treats the entire region as one homogeneous group, which fails to account for many complexities and the heterogeneity of the space. Since the distribution of fractals is not uniform, they are similar to cities, and thus the math associated with them (specifically fractal dimension) is a better measurement tool for urban spaces. [2]

Using the box-counting method discussed above, mathematicians and urban planners can find the fractal dimension of cities. Large and dense spaces have fractal dimensions closer to 2 (like New York City at 1.7014) while smaller, less populated areas have dimensions closer to 1 (like Omaha, Nebraska at 1.2778). By standardizing a way to measure the irregularity of cities, it becomes much easier to predict population growth and make comparisons. Some researchers have even found a positive correlation between the log of a population and the fractal dimension, demonstrating that population growth can be described by fractal geometry (Figure 7). However, seeing as this is a new field, there are not yet classification methods to sort and draw conclusions based solely on the fractal dimensions of cities, but those discoveries and categorizations will likely come soon. [8]

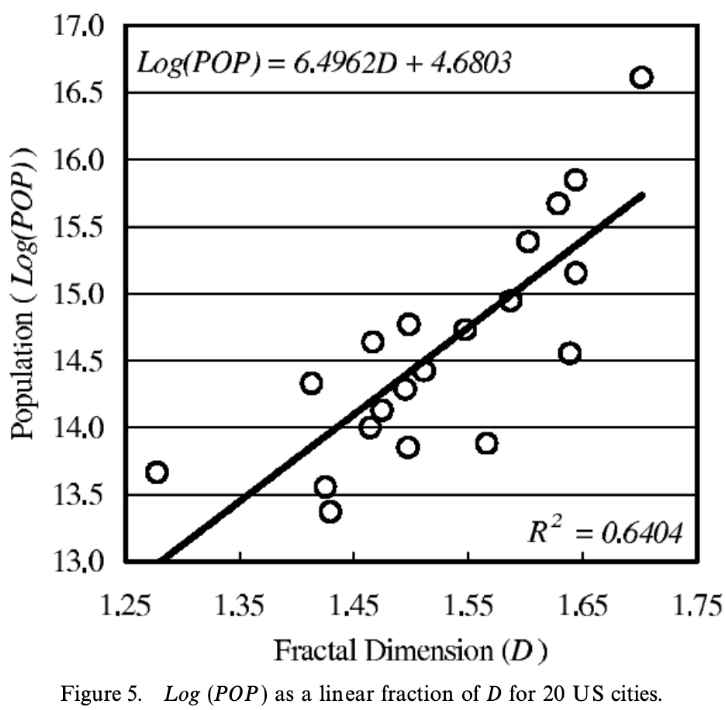

Figure 7. The (log) population versus the fractal dimension of 20 US cities [8]

4. Fractals and Faith

The final dimension of fractal geometry within cities pertains to architectural design, particularly in religious contexts. Cathedrals from the Gothic, Renaissance, and Baroque periods feature a distinctive arrangement of self-similar spires. Similarly, Hindu temples adhere to this principle, incorporating self-similar domes that progressively increase in size from the entrance to the main tower. In either case, the fractal elements are used to show a connection to heaven and emphasize the significance of certain parts of the building.

While not the focus of this study, my research also led me to sources arguing that fractals are proof of the existence of a creator, as they are called the "geometry of eternity." Fractals give humans an understandable way to organize apparent chaos, an idea that some speculate would only be possible with a divine entity. Regardless of whether one believes a creator is their cause, fractals (and their many applications) are proof of the importance of infinity in daily life and in shaping the world around us. [7] [9]

5. Reflection

In completing the talk, I was pleased with the response and discussion afterward, as the argument of what classifies a fractal and whether new, block-based, cities can be described with fractal geometry was fascinating. It was interesting to hear about how different students in the class understood the analytical side of the talk and applied it in ways I had not yet thought of. If given the chance to repeat the talk, I would focus more on making sure everyone understands how fractal dimensions are calculated, as by talking quickly I left a few classmates confused, and fractal dimensions were a central theme in my discussion. Regardless, the connections the class made to walkable cities and philosophical arguments piqued my interest, and I am considering continuing along that path for future presentations.

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