The Philosophy of Geometry and Descartes' Algebra

Henry Stratakis-Allen
Math 400: Presentation 1

Descartes

Voltaire

Kant

Gauss

1600

1700

1800

Euler

Diderot

Philosophy of Forms: Two Schools

(with respect only to the existence per se of geometric forms)

Descartes:

‘Mathematical and logical axioms are held to be apprehended intuitively’, ‘logical axioms are necessarily and universally true’, ‘mathematical truth is necessary and universal’, and ‘all innate ideas clearly and distinctly perceived are always and inevitably true’. [Gibson, 169]

René Descartes (1596, Touraine - 1650, Stockholm)

Diderot:

Suspended axiom systems as “concepts with no foundations in nature … may be compared to those Northern forests where the trees have no roots. It needs nothing more than a gust of wind, or some trivial event, to bring down a whole forest of trees – and of ideas … so long as something exists only in the mind, it remains there as an opinion”. [Thoughts on the Interpretation of Nature, 39]

Denis Diderot (1713, Langres - 1784, Paris)

La Mettrie: “Let some one attach a banner to this bit of wood and another banner to another similar object; let the first be known by the symbol 1, and the second by the symbol or number 2 … as soon as one figure seems equal to another in its numerical sign, man will decide without difficulty that they are two different bodies, that 1 + 1 make 2, and 2 + 2 make 4, etc … all this knowledge, with which vanity fills the balloon-like brains of our proud pedants, is therefore but a huge mass of words and figures, which form in the brain all the marks by which we distinguish and recall objects”. [Man a Machine, 106]

Julien Offray de La Mettrie (1709, Saint-Malo - 1751, Berlin)

D'Holbach: "The universe … presents only matter and motion: the whole offers to our contemplation nothing but an immense, an uninterrupted succession of causes and effects". [The System of Nature, 15]

D'Alembert: "All our direct knowledge can be reduced to what we receive through our senses; whence it follows that we owe all our ideas to our sensations … after having reigned for a long time, the system of **innate ideas** still retains some partisans—so great are the difficulties hindering the return of truth, once prejudice or sophism has routed it from its proper place". [Preliminary Discourse, i-xlv]

Paul Heinrich Dietrich, Baron d'Holbach (1723, Edesheim - 1789, Paris); Jean-Baptiste le Rond d'Alembert (1717, Paris - 1783, Paris)

Euler:

"The general idea which comprehends all is formed only by abstraction … the fault which these philosophers are ever finding with geometricians, for employing themselves about abstractions merely, is therefor groundless, as all other sciences principally turn on general notions, which are no more real than the objects of geometry', and 'the very merit of each science is so much the greater, as it extends to notions more general, that is to say, more abstract".
[Letters to a German Princess, 32]

Leonhard Euler (1707, Basel - 1783, Saint Petersburg)

Gödel:

"Mathematics describes a non-sensual reality, which exists independently both of the acts and the dispositions of the human mind and is only perceived, and probably perceived very incompletely, by the human mind. This view is rather unpopular among mathematicians, there exist however some great mathematicians who have adhered to it".
[Unpublished Philosophical Essays III, 147]

Kurt Friedrich Gödel (1906, Brünn - 1978, Princeton)

Philosophy of Forms: Two Schools

Descartes

Gödel

Euler

Diderot

D'Alembert

D'Holbach

La Mettrie

(with respect only to the existence per se of geometric forms)

Kant:

"Geometrical principles are always apodeictic, that is, united with the consciousness of their necessity, as: 'space has only three dimensions'. But propositions of this kind cannot be empirical judgements, nor conclusions from them. How can an external intuition anterior to objects themselves, and in which our conception of objects can be determined à priori, exist in the human mind?". [Critique of Pure Reason, p. 3]

Kant's answer: because space (as a Euclidean continuum) is fundamental to the human mind:

"Space is nothing else than the form of all phenomena of the external sense, that is, the subjective condition of the sensibility, under which alone external intuition is possible". [Critique of Pure Reason, p. 4]

Immanuel Kant (1724, Königsberg - 1978, Königsberg)

Kantian orthodoxy (early 1800s):

- Euclidean geometry is 'inherent in the structure of our mind' [Greenberg, 245]
- The Euclidean geometric paradigm as a 'necessity of thought'
- Space as an infinite continuum that does not 'exist per se' outside of the human mind
- Humans have an à priori conception of (Euclidean) space and time

> "Hence it follows that an à priori intuition (which is not empirical) lies at the root of all our conceptions of space. Thus, moreover, the principles of geometry—for example, that "in a triangle, two sides together are greater than the third," are never deduced from general conceptions of line and triangle, but from intuition, and this à priori, with apodeictic certainty". [Critique of Pure Reason, p. 2]

- So, preconceptions about space impact one's understanding of forms

Descartes' Algebra of Lengths

- Motivated by frustration with traditional assumptions about complex polynomials
- The goal is to establish a corollary (bijection) between a Euclidean field and the real numbers using the Euclidean axioms
- Undefined terms: 'point', 'line', 'between', 'incident'

Proposition 2.1. *If the points on m all belong to one of two disjoint sets A_1 and A_2, which are such that whenever two points, say P, Q, belong to the same set A_i, then all points between P and Q also belong to A_i, then there exists a unique point X which lies between each point in $A_1 \setminus \{X\}$ and each point in $A_2 \setminus \{X\}$.*

Fig. A

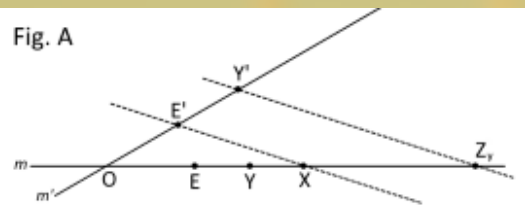


Fig. B

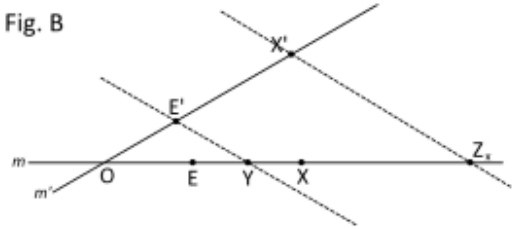
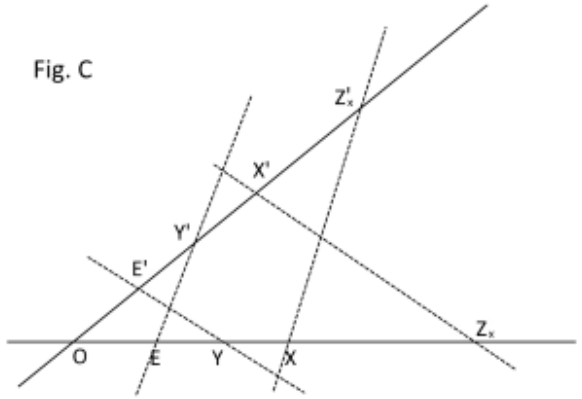
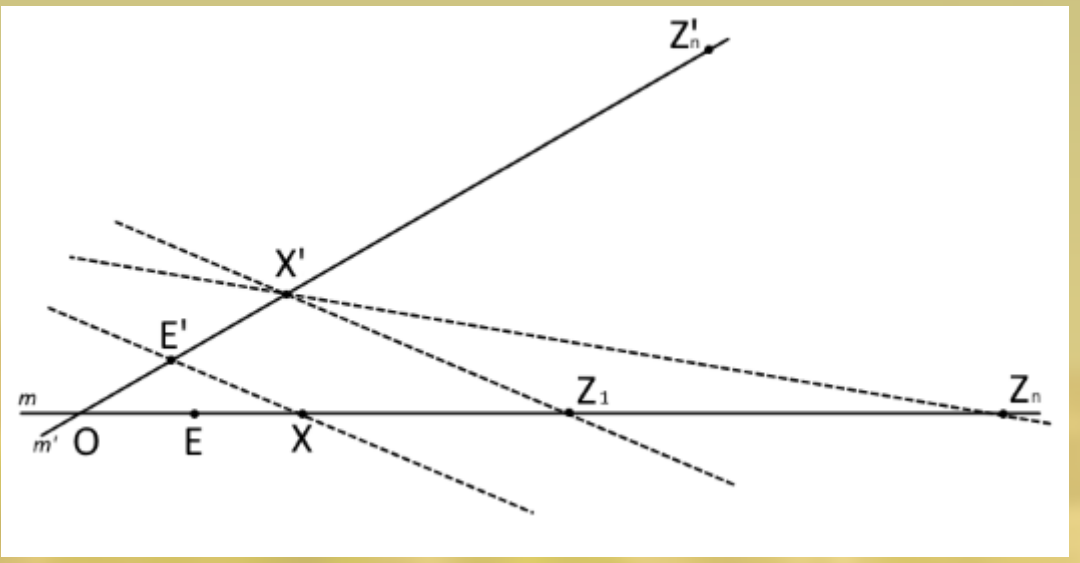
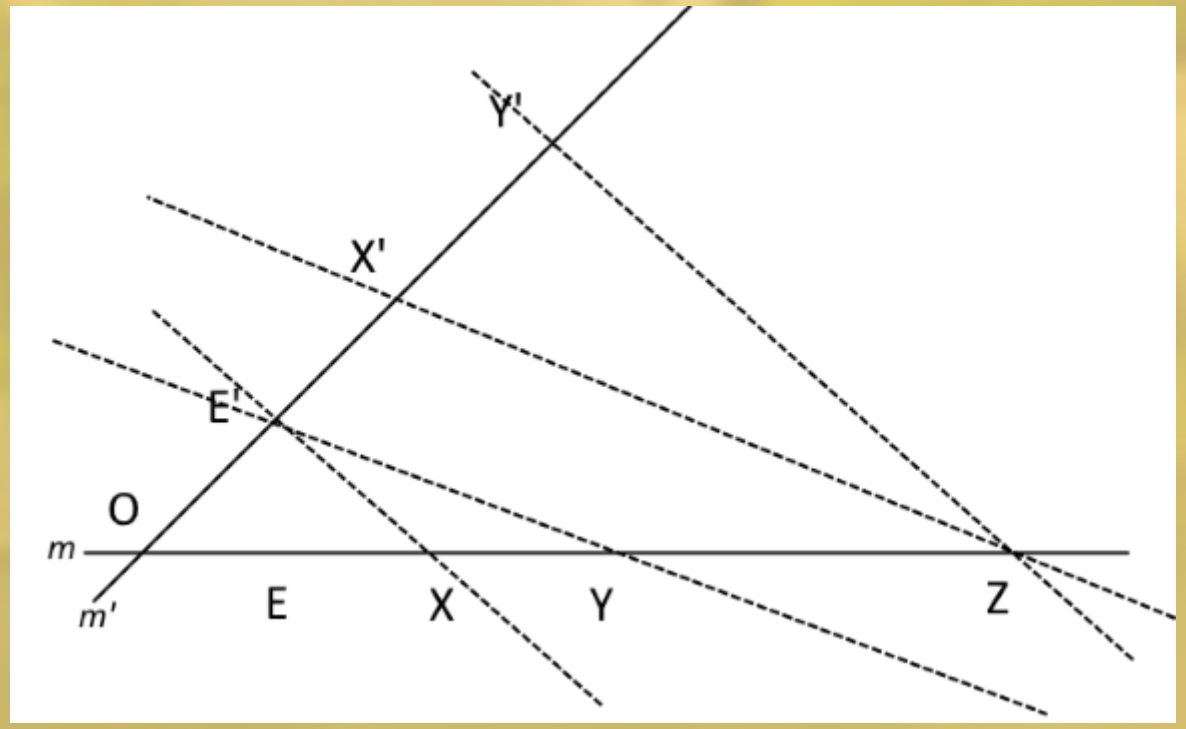
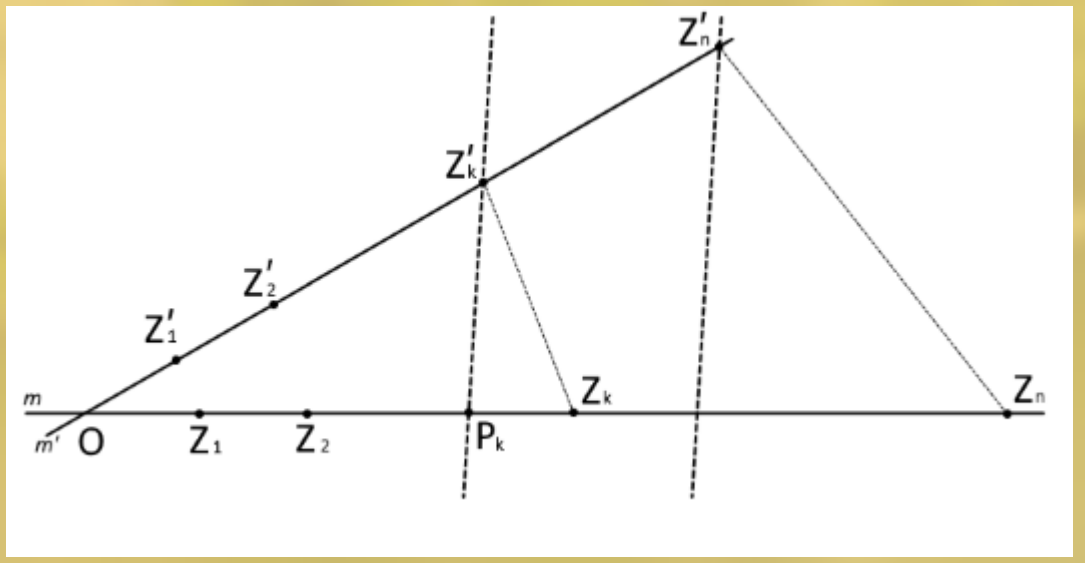


Fig. C









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Descartes' method of coordinates "revolutionized the treatment of geometrical problems and provided the appropriate instrument for the description of the phenomena of motion in modern physics". [Greenberg, 34]

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