Henry Stratakis-Allen
Math 400 – Presentation 1 Essay
The Philosophy of Geometry and Descartes' Algebra

Abstract

In this paper, I will briefly survey the origins of two major schools of western epistemology with respect to their mathematical implications; I will characterize these two schools as the so-called Aristotelian-materialist school, which advocates that mathematical objects (such as numbers, axioms, shapes, structures, etc.) are non-real entities invented by humans to describe nature (which is the only source of real knowledge), and the so-called Platonistic-realistic school, which advocates that mathematical objects exist *per se* independent of human intellect. I will focus on the origin of these representations of modern western epistemology in the Enlightenment with a particular emphasis on how mathematicians and philosophers approached the topic from different perspectives. This paper will demonstrate how mathematics and philosophy have been historically intertwined and how the intersection of the two disciplines has influenced the development of topical cultural issues such as the theory of knowledge.

Introduction

Many of the philosophical and political issues that shape western thought today are fundamentally inter-disciplinary; one such issue is the problem of modern epistemology, which draws as much from traditions of Enlightenment philosophy as it does from the traditions of nineteenth-century mathematics. The intellectual friction that exists in the modern west is largely the result of competing epistemological assumptions that structure how individuals grasp knowledge and approach the search for truth. These competing assumptions are illustrated here by two divergent schools of thought that emerged in the eighteenth-century with respect to the epistemology of mathematics.

I will first introduce the historical context of the Enlightenment by discussing several key figures in the development of modern western philosophies of mathematics. Second, I will discuss the mathematical framework against which many Enlightenment philosophers reacted: the philosophy of Rene Descartes. Third, I will introduce the philosophy of Denis Diderot as the archetype of Enlightenment philosophy of mathematics in France and as representing the philosophical framework that the French philosophers hoped would replace the Cartesian framework. Fourth, I will characterize French philosophy of mathematics in general using several examples. Fifth, I will introduce Euler's mathematical philosophy as a reaction against the French thinkers to demonstrate the development of a counter-Enlightenment epistemology of mathematics. Last, these developments will be used to provide context for the continued schism in mathematical philosophy among mathematicians, which will be described in reference to the controversial philosophical views of Kurt Godel. I will tie these sections together by demonstrating the philosophical implications of the systems I describe with reference to an example of a mathematical object that was discussed by both Descartes and Euler: the Exterior Angle Theorem in Euclidean geometry. The goal of this paper is to demonstrate the existence and potential volatility of relationships between mathematics and philosophy and to indicate the usefulness of studying these inter-disciplinary topics for the purpose of understanding better the intellectual dilemmas of the modern west.

In the next few sections, I will describe the epistemological attitudes towards mathematics of Descartes, Diderot, d'Alembert, d'Holbach, La Mettrie, Euler, and Godel. These sections will be proceeded by a paragraph discussing the conclusions of this paper, further reading and possible directions for future research, and comments regarding the original plan for my in-class presentation.

'Schism between Theory and Practice'

The mathematician Leonhard Euler nurtured rivalries with several of his colleagues across western Europe. The dilemmas that these rivalries produced have sometimes been called ‘a symbol for the schism between theory and practice in the 18th century and beyond’. In 1747, for example, d’Alembert sent an angry letter to Lagrange complaining about Euler’s ‘incompetence in metaphysical questions’ because of Euler’s opposition to his monadic philosophy; coincidentally (or perhaps not), this was the same year that Euler disproved d’Alembert’s theory that logarithms of negative numbers are real. In 1748, Euler attempted to block the election of the *philosophe* Julien Offray de La Mettrie to the Berlin Academy of Sciences on account of his militant atheism. And in 1752, Voltaire published a pamphlet that contained several criticisms of Euler’s philosophy of mathematics. In order to understand the tense philosophical and political environment that generated these disputes between a mathematician and these Enlightenment philosophers, it is necessary to reconstruct the philosophical framework that the French *philosophes* were reacting against: namely, the philosophy of Rene Descartes, which became increasingly controversial throughout the eighteenth century despite representing the basis of modern mathematics.

Descartes’ Mathematical Philosophy

Descartes had argued that ‘mathematical and logical axioms are held to be apprehended intuitively’, ‘logical axioms are necessarily and universally true’, ‘mathematical truth is necessary and universal’, and ‘all innate ideas clearly and distinctly perceived are always and inevitably true’. Likewise, ‘all knowledge, for Descartes, is of innate ideas … guaranteed by God’. Descartes believed that the origins of truth are to be found in received knowledge, planted in the mind, although he also admitted that, when one exhausts deduction from within the mind to the external world without reaching a result, ‘I know of no other device than to look about forthwith for some kind of experiment’. So, Descartes’ epistemology proceeds as follows: men access necessary, universal, and eternal truths through intuition, which appear in the mind through the intercession of God; the scientific method consists of movement from these simple, general truths to more complex knowledge, sometimes supplemented by experiment. Knowledge in this system is ultimately internal: universal truths, which descend to mankind from an external non-sensual reality, are retrieved from within the human mind, and the philosopher may choose to proceed from there to the external world. Consider the example of the Exterior Angle Theorem (‘EA theorem’) in Euclidean geometry (the EA theorem also holds in neutral geometry, but it is historically convenient to assume the theorem in Euclidean space instead). How would an advocate of Descartes’ philosophy approach the question of the EA theorem’s existence? To start, such an individual would likely argue that the system of Euclidean geometry exists by itself, independent of human thought or action. Like all mathematical objects, Euclidean space and its axioms/theorems exist in the non-sensual reality from which human beings extract truth through reflection. To this extent, it is obvious that human beings have made certain discoveries about Euclidean space over time and that our knowledge of these systems has grown and changed. As such, the EA theorem would be considered one element of the system (or set) of theorems that hold in Euclidean geometry: it exists in itself as an evident property of Euclidean space that we have discovered. What precisely it means for such an item to really ‘exist’ is not necessarily obvious or important: it is only important that it exists by itself in some non-physical reality, a world of ideas created by God.

It was Descartes’ theory of innate ideas – his epistemological doctrine – against which the *philosophes* of the eighteenth century reacted. This reaction should be understood in the context of a more general rejection of systematic axiom systems by the *philosophes*, which Denis Diderot’s philosophy exemplifies.

Diderot’s Mathematical Philosophy

Many French philosophers of the eighteenth century viewed axiom systems with distrust. For them, axiom systems were artificial paradigms of thought invented by men to facilitate the study of nature. While they were often useful, they were ultimately dispensable, particularly if they represented a barrier to free thought. Some philosophers, for example, believed that mathematical axiom systems were not timeless: Diderot argued that all scientific fields were governed by life-cycles. For Diderot, systems of scientific thought are ‘requiring experimental verification’ and, more dramatically, ‘are propped up by nothing more than vague ideas, mild suspicions, and deceptive analogies’. Diderot believed that all such systems were bounded by their finite productive potential and obey a generic life cycle: the dawning of a new science is followed by the flocking of philosophers to its school as a means of achieving fame and honors; ‘as its boundaries are stretched even further, the esteem in which it is held diminishes’; and then ‘the crowd thins out; no one sets off for a land where fortunes are rarer and harder to make’; its prestige fades, the antiquated philosophers who cling to it slowly realize that attempts at further progress are futile, and eventually the field becomes extinct. The works of its philosophers are recognized as ‘awesome pictures of the might and resources of the men who built them’, as monuments, like the Egyptian pyramids, ‘which do honor to humankind’.

Clearly, Diderot viewed scientific methodologies as time-limited human constructs, built to service a particular aim and facilitate a particular field of study. They have finite lifespans, and they eventually all die, when ‘they cease to instruct or delight’. The example of the EA theorem in Euclidean geometry is extremely instructive in this context because Diderot wrote specifically about geometry, which in his time consisted only of Euclidean geometry and the mathematical dilemmas that it represented prior to the discovery of the consistency of non-Euclidean geometries. Diderot thought that (Euclidean) geometry was nearing the end of its lifespan during his own life, while experimental science was at its dawn. Diderot predicted that ‘within the next hundred years, there will hardly be three great geometricians in Europe … [eighteenth-century geometry] will stand like the Pillars of Hercules and no one will pass beyond’. Thus, like any other scientific system, the axioms of geometry were aging and nearing the end of their usefulness. It is also evident that Diderot viewed Euclidean geometry and empirical science as disjoint fields of study. Thus, Euclidean geometry represented to Diderot a collection of hypotheses about space that were invented by humankind and did not represent truth in themselves. Diderot rejected ‘systematic, mathematical philosophy dear to the rationalist school, which he accuses of being preoccupied with theorizing and unconcerned with experimental facts’. Likewise, he considered such axiom systems “concepts with no foundations in nature … [which] may be compared to those Northern forests where the trees have no roots. It needs nothing more than a gust of wind, or some trivial event, to bring down a whole forest of trees – and of ideas … so long as something exists only in the mind, it remains there as an opinion”. Diderot considered Euclidean geometry one of these ‘Northern forests’, liable to be destroyed at any moment by some new discovery, and which had no basis in facts, which only come from nature and not from the human mind.

French Mathematical Philosophy in General

Many of Diderot’s views were widely held among the *philosophes*, for example by his colleague Jean le Rond d’Alembert. While d’Alembert did not agree with Diderot about the life-cycles of sciences, he did share Diderot’s materialistic epistemology. He that ‘all our direct knowledge can be reduced to what we receive through our senses; whence it follows that we owe all our ideas to our sensations … after having reigned for a long time, the system of innate ideas still retains some partisans – so great are the difficulties hindering the return of truth, once prejudice or sophism has routed it from its proper place’. This quote exemplifies the reaction against Cartesian epistemology in general, through which the *philosophes* sought to replace internal, eternal mathematical truth with a knowledge system based entirely on sensation.

In the shadow of the mainstream Enlightenment at Paris (here represented by Diderot and his close colleagues), several other *philosophes* developed even more radical views on the source and nature of true knowledge; these views have generally been identified with the origins of modern materialism. The Baron d'Holbach, for example, wrote that 'the universe … presents only matter and motion: the whole offers to our contemplation nothing but an immense, an uninterrupted succession of causes and effects'. This is a clear statement of the belief that nothing in the universe 'exists' that cannot be described by matter or motion, i.e. by some attribute that is comprehended by or in relation to the human senses. Julien de La Mettrie, working more closely with mathematical topics, argued that all of mathematics is nothing more than a system of signs used to reference physical objects: 'let some one attach a banner to this bit of wood and another banner to another similar object; let the first be known by the symbol 1, and the second by the symbol or number 2 … as soon as one figure seems equal to another in its numerical sign, man will decide without difficult that they are two different bodies, that 1 + 1 make 2, and 2 + 2 make 4, etc… all this knowledge, with which vanity fills the balloon-like brains of our proud pedants, is therefore but a huge mass of words and figures, which form in the brain all the marks by which we distinguish and recall objects'. Thus, La Mettrie rejected the actual existence not only of mathematical theorems, axioms, and constructions, but also of numbers and even equivalence relations. Returning to the example of the EA theorem, the implications of La Mettrie's philosophy are somewhat self-evident. Like Diderot, he would have considered Euclidean geometry to be a system, or thought paradigm, of human invention. For him, Euclidean geometry had no basis or connection to actual physical space; it is rather a set of principles that make reference to external sensual experience but do not represent external sensual experience (i.e. truth) in themselves. Likewise, La Mettrie would have considered the EA theorem to be nothing more than a set of signs intended by construction to represent the objects of sensual experience. He would not have considered it a 'true' theorem in the sense of representing some truth about reality because he would have considered Euclidean geometry itself to be an invented system. Thus, La Mettrie would have rejected both the actual existence of the EA theorem and its attempt to represent a truth of the universe, which *philosophes* such as d'Holbach considered to consist only of physical things.

Euler: A Reaction against French Philosophy

Unsurprisingly, the 'schism between theory and practice' is visible in these sources. La Mettrie referred to some of his contemporaries angrily, referencing 'the balloon-like brains of our proud pedants'. D'Alembert mentioned that 'the system of innate ideas still retains some partisans'. While Voltaire and several other *philosophes* dominated eighteenth-century intellectual life in Europe, there remained an indefatigable party of realists who continued to defend, as d'Alembert says, the 'system of innate ideas', which usually advocates the existence of mathematical truth and the *per se* existence of certain mathematical theorems, axioms, constructions, etc. Euler may be counted among their number, as he defended the field of pure mathematics (and geometry in particular) against the charges of *philosophes* who argued that its propositions reference nothing that actually exists. In particular, he wrote that 'the general idea which comprehends all is formed only by abstraction … the fault which these philosophers are ever finding with geometricians, for employing themselves about abstractions merely, is therefor groundless, as all other sciences principally turn on general notions, which are no more real than the objects of geometry … the very merit of each science is so much the greater, as it extends to notions more general, that is to say, more abstract'. Likewise, for Euler 'the difference which [certain philosophers] establish between objects formed by abstraction and real objects' creates an epistemology in which 'no conclusion, and no reasoning whatever, could subsist', meaning that the existence of abstract mathematical constructions (such as triangles, the EA theorem, etc.) must admitted if any truth is to be acquired. The implications of this philosophy for the existence of the EA theorem are clear. Euler rejected the distinction between real and nonreal ideas as being between ideas 'formed by abstraction' and ideas formed by sensory observation of nature. Thus, it is likely that Euler would have rejected the position of Diderot that Euclidean geometry was an invention of mankind, or at least the assumption that Euclidean geometry is a nonreal system simply due to its abstract nature. Similarly, Euler would have rejected the assumption that the EA theorem is nonreal simply because it is abstract, general, and does not reference nature or sensory experience directly. It is similarly possible that Euler would have gone as far as to advocate the real existence of such propositions, which seems to be the direction in which his philosophy leans.

While Euler's platonic philosophy was not popular in his own time, several of his greatest mathematical successors held similar views. Hermite and probably Gauss could be included, but none produced philosophical work as deep as Kurt Godel's. His philosophy represents a modern, mature formulation of classical platonic philosophy of mathematics and mathematical epistemology; he wrote that 'mathematics describes a non-sensual reality, which exists independently both of the acts and the dispositions of the human mind and is only perceived, and probably perceived very incompletely, by the human mind. This view is rather unpopular among mathematicians, there exist however some great mathematicians who have adhered to it'. Thus, Godel advocates the existence not only of specific mathematical constructs (such as the real line, certain geometric spaces, and geometrical objects) but also of a whole universe of mathematics, including fundamental mathematical truths (axioms) as well as theorems and relations, all drawn from a distinct 'non-sensual' sphere, which we may perceive through reflection. Clearly, the EA theorem represents one item in the set of propositions that are true in Euclidean geometry, which we access through our intelligence, while that set of propositions itself really exists in the 'non-sensual sphere' of mathematical truths. These ideas have generally become controversial, as Godel himself indicates, and yet continue to represent the philosophical opinions of several of the modern world's greatest mathematicians. Thus, the so-called 'schism between theory and practice', i.e. between philosophical theory and mathematical practice, remains evident to this day (with popular Aristotelian-materialistic views representing philosophical theory and Godel's Neoplatonist realism representing one form of mathematical practice).

Conclusion

In my original presentation, I had intended to discuss how Kant's philosophy bridges the gap between the mathematics of the Enlightenment and the mathematics of the modern world, and then circle back to discuss the impact of Descartes' algebra on mathematical philosophy and why his 'system of innate ideas' became fundamental to the development of modern mathematics. Indeed, this topic indicates many possible directions for further research, for example how the Enlightenment French philosophers would react to modern discoveries in physics and astronomy that seem to indicate the physical existence of entities that are impossible, or very difficult, to observe directly. However, I realize now that it would have been suitable to end the presentation by mentioning Godel. Thus, I will conclude that the mathematical philosophy of the French Enlightenment was more radical in some circles than has sometimes been admitted, and that the greatest philosophers of the French Enlightenment advocated mathematical doctrines that were unintuitive and even unacceptable to several of history's greatest mathematicians. This sufficiently demonstrates the potential volatility of the relationship between mathematics and philosophy. The *per se* existence of certain mathematical elements remains controversial even today, although the question represents only one facet of the perpetual struggle between advocates of Platonistic and Aristotelian epistemologies. This controversy is heightened by the immediate implications that the theory of knowledge has in topics of and relating to the sciences, for example in the importance one places in theory versus practice.

Bibliography

• Bos, Henk. *Redefining Geometrical Exactness*. New York: Springer-Verlag, 2001.

• Calinger, Ronald. *Leonhard Euler: Mathematical Genius in the Enlightenment*. Princeton: Princeton University Press, 2016.

• Gibson, A. Boyce. *The Philosophy of Descartes*. London: Methuen & Co., ltd., 1932.

• Greenberg, Marvin J. *Euclidean and Non-Euclidean Geometries*. 4th ed. New York: W.H. Freeman, 2008.

• Torretti, Roberto. *Philosophy of Geometry from Riemann to Poincaré*. Dordrecht-Holland: D. Reidel Pub. Co., 1978.

• Van Heijenoort, Jean. *From Frege to Gödel; a Source Book in Mathematical Logic, 1879-1931*. Cambridge: Harvard University Press, 1967.

• D'Holbach, Paul Heinrich. *The System of Nature*. Boston: J. P. Mendum, 1868.

• Diderot, Denis, D. J. Adams, and Denis Diderot. *Thoughts on the Interpretation of Nature: and Other Philosophical Works*. Manchester: Clinamen, 1999.

• Diderot, Denis, and Lester G. Crocker. *Selected Writings*. New York: Macmillan, 1966.

• Diderot, Denis, and Stephen J. Gendzier. *Denis Diderot's The Encyclopedia: Selections*. 1st Harper Torchbooks ed. New York: Harper & Row, 1967. Accessed via quod.lib.umich.edu/d/did.

• Euler, Leonhard, David Brewster, and John Griscom. *Letters of Euler on Different Subjects in Natural Philosophy: Addressed to a German Princess*. New York: J. & J. Harper, 1833.

• Gödel, Kurt. *Unpublished Philosophical Essays* III. Basel: Birkhäuser Verlag, 1995.

• La Mettrie, Julien. *Man a Machine*. Chicago: The Open Court Publishing Co., 1912.