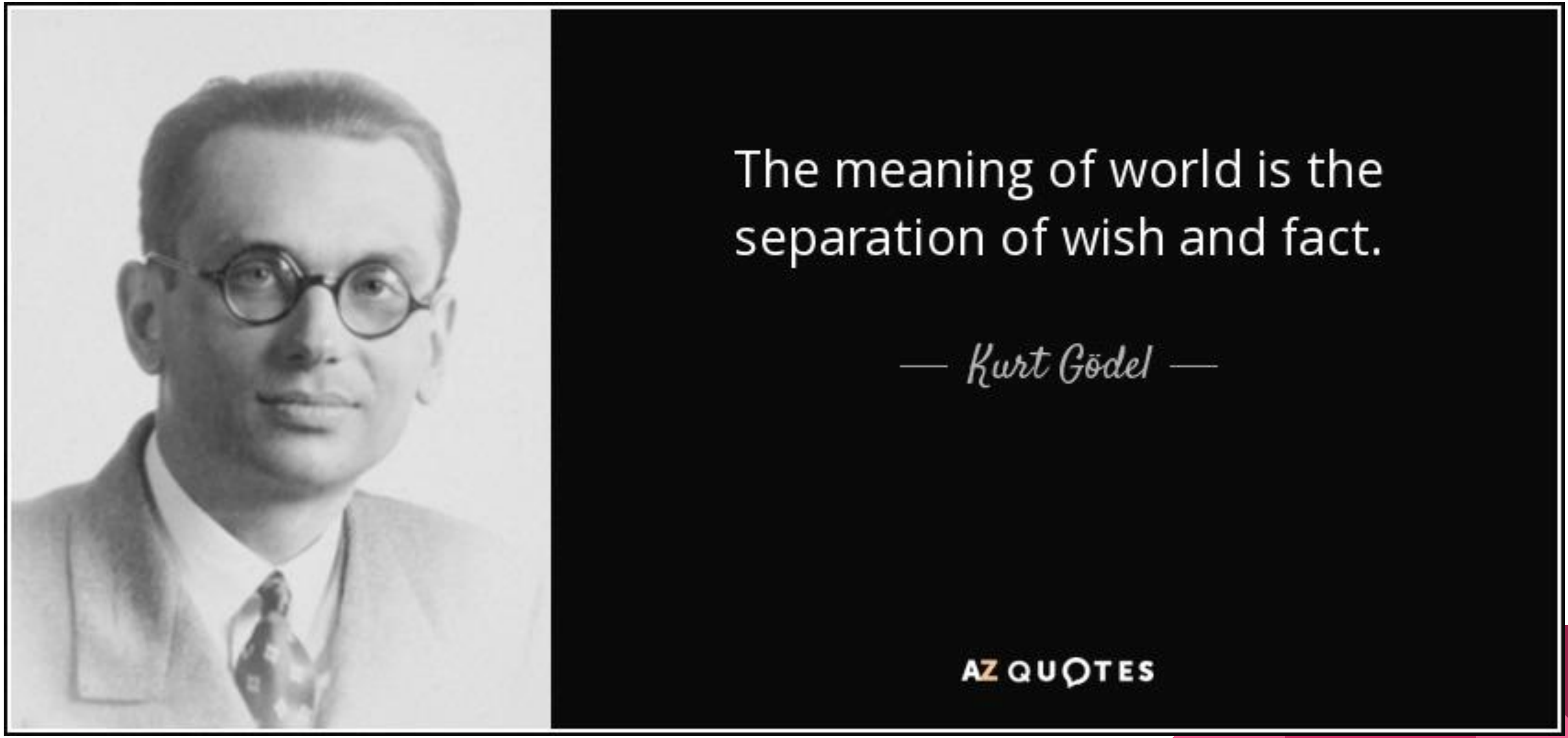
Godel's First Incompleteness Theorem

An attempt to explain parts of a simplified proof of Godel's First Incompleteness Theorem

Godel’s First Incompleteness Theorem

Open Question: Can we ever know everything?


Wir mussen wissen. Wir werden wissen. We must know. We will know. Inscribed on his tomb in Gilttingen.
— David Hilbert —
AZ QUOTES


The meaning of world is the separation of wish and fact.
— Kurt Gödel —
AZ QUOTES

Terminologies: Godel’s First Incompleteness Theorem

1. Formal System : e.g., Euclidean Geometry (f), ZFC, Q

2. Completeness : all statements or their negation are provable within the system

3. Consistency : no contradiction

4. Decidability (more involved in 2nd Inco. Thm. and Godel’s other thms)

ONLY consistent formal systems

Principle of Explosion (not good): If a formal system is inconsistent with one example of contradiction, everything (statements and their negations) would be provable.

Godel’s First Incompleteness Theorem

(More Precise) Statement: Any consistent formal system F within which a certain amount of elementary arithmetic can be carried out is incomplete;

i.e., there are statements of the language of F which can neither be proved nor disproved in F.

Preliminaries: Concepts and Lemmas

1. Ω-consistency/1-consistency (Godel's formalizations of "consistency"):
   - Ω-consistency is natural consistency; 1-consistency is restricted Ω-consistency applied only to certain formulas. Godel's F is at least 1-consistent.
2. Representability:
   - Strongly Representable (S.R)/Weakly Representable (W.R)
3. Godel Numbering: Formal Language as Arithmetics
4. Diagonalization Lemma/"Self-Referencing"

Preliminaries: Godel Numbering

Expressing the 12 necessary symbols in a basic formal system F

- We use these symbols to express axioms, construct proofs, and talk about statements themselves

Constant sign	Gödel number	Usual Meaning
~	1	not
∨	2	or
⊃	3	if...then...
∃	4	there is an...
=	5	equals
0	6	zero
s	7	the successor of
(	8	punctuation mark
)	9	punctuation mark
,	10	punctuation mark
+	11	plus
×	12	times

Preliminaries: Diagonalization Lemma

Statement: Let A(x) be an arbitrary formula of the language of F with only one free variable, then a sentence D can be mechanically constructed such that:

$$F \vdash D \leftrightarrow A(\ulcorner D \urcorner).$$

Result of the Thm:

To complete the proof, the Diagonalization Lemma is applied to the negated provability predicate $\neg Prov_F(x)$: this gives a sentence G_F such that

$$F \vdash G_F \leftrightarrow \neg Prov_F(\ulcorner G_F \urcorner).$$

Gobel # of formula G_F

$Prov_F(\ulcorner G_F \urcorner)$ denotes :

$\exists x$ s.t. x is the Godel # of a proof of a formula G_F in F.

$\neg Prov_F(\ulcorner G_F \urcorner)$ denotes:

No x exists s.t. x is the Godel # of a proof of a formula G_F in F.

$\Leftrightarrow$ G_F is unprovable in F.

Proof:

To complete the proof, the Diagonalization Lemma is applied to the negated provability predicate $\neg Prov_F(x)$: this gives a sentence G_F such that

$$F \vdash G_F \leftrightarrow \neg Prov_F(\ulcorner G_F \urcorner).$$

Lemma: there's a G_F that would establish the following:

$G_F \leftrightarrow G_F$ is unprovable in F.

Proof: sub(a,b,c), a Godel Number

sub(a, b, c) :
1 3 2

1.) find proposition with Godel # a

2.) find position of the symbol where the Godel # is c.

3.) Replace C(s) with b.

4.) Above steps create a proposition, and its Godel # we define to be sub(a,b,c).

Proof: sub(y,y,17) and sub(n,n,17)

Formula 1:

"The formula with Godel # sub(y,y,17) can't be proven"

don't need to care abt what this is.

Godel # of Formula 1 = n.

Formula 2 (G_F):

"The formula with Godel # sub(n,n,17) can't be proven"

Godel # of Formula 2: sub(n,n,17).

Proof: Is the sentence G provable?

Assume G is provable, some sequence of formula exists that proves that formula with Godel # sub(n,n,17), but that is the opposite of G, meaning both G and neg(G). By consistency, this is a contradiction, so G is unprovable.

Although G is unprovable, it is true: G says G is unprovable, but this is the conclusion we just reached above.

So, we have something that's true and unprovable in F…..

To complete the proof, the Diagonalization Lemma is applied to the negated provability predicate $\neg Prov_F(x)$: this gives a sentence G_F such that

$$F \vdash G_F \leftrightarrow \neg Prov_F(\ulcorner G_F \urcorner).$$

References

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