Zeno's Paradox: Context & Solutions(?)

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Abstract

In this paper, I examine Zeno's paradox, "the Dichotomy," by presenting the paradox and several proposed mathematical solutions to it. I begin by providing historical context for the paradox by discussing the Eleatic School in ancient Greek philosophy, and then I present the paradox itself which argues something in motion cannot reach its final endpoint because it must go halfway before reaching the endpoint, and when we repeatedly go the halfway remaining we never reach the endpoint. I present a series of proposed solutions to this problem: the Formal-Logical Solution, the Finite Bounds Solution, the Infinite Sum Solution, and the Modeling Solution. The first argues formal logic provides allows us to formulate a better argument against the setup of the paradox than the argument the paradox itself provides. The second argues that Zeno is wrong to assume that infinitely many distinct journeys would take an infinite amount of time because the length of time it takes is contained in a bounded interval. The third argues similarly that modern analysis allows us to show that an infinite sum of all these halfway trips converges to the finite value that represents the endpoint. The fourth proposes that Zeno's paradox models motion incorrectly, even if its mathematics is consistent. In the process, I argue that the strongest interpretation of Zeno's paradox does not presuppose that infinitely many trips would take an infinite amount of time and so does not imply that the infinite series does not diverge. As a result, I suggest that the strongest solution I survey is the modeling solution.

Introduction

Zeno's most famous paradox is "the Dichotomy", which I will refer to simply as Zeno's Paradox. The paradox provides a frustratingly simple argument for a shocking conclusion: Motion is impossible. Because the paradox is very quantitative in nature, it is often analyzed mathematically. Indeed, it is sometimes claimed that modern analysis and calculus can solve the paradox. Yet, against this, there is still a camp within philosophy and intellectual history that maintains that mathematics cannot provide a solution to Zeno's Paradox.[1]

Here, then, my goal is not to try to solve the paradox using mathematics, though a specific interpretation of the paradox certainly can be solved mathematically. My primary goal is to create a *dialogue* between philosophy and mathematics using this paradox. I would like to make a *two-sided* connection between philosophy and mathematics, letting the two disciplines inform one another, in a push and pull. Accordingly, I will explain what Zeno is attempting to do philosophically with his paradox(es). Then, I will provide a series of mathematical solutions and provide what I take to be good replies Zeno could make to them. In many cases, in my view, these replies – though they require

[1] See, e.g., Papa-Grimaldi, "Why Mathematical Solutions of Zeno's Paradox Miss the Point."

a different, stronger interpretation of the paradox differently than often offered – sufficiently respond to the mathematical objections to Zeno's construction. The mathematical solution then needs to be improved in order to answer Zeno's objection, and this paper aims to create this back-and-forth.

I. Background to Zeno's Paradox

Zeno of Elea lived from around 495 BCE to 430 BCE, and he was a member of what is called the "Eleatic School", based in the Elea in ancient Greece.[2] The Eleatic School was founded by a philosopher named Parmenides, and his thought frames the Zeno's paradoxes.

The argument made in Zeno's paradoxes is generally negative: A contradiction which arises if we accept the possibility of motion, diversity or plurality, and the like, and so we should reject their possibility. These arguments, however, play a positive role in the general philosophical framework established by Parmenides. Before we turn to Zeno's Paradox, then, we should first figure out what exactly he is trying to accomplish by creating a paradox. Though there are multiple interpretations of Parmenides today,[3] I will provide a basic overview of the one that seems to be the standard reading in the history of philosophy. Parmenides concisely states the core of his argument and philosophy in several places, one of which is this:

> Come now, I will tell thee ... the only two ways of inquiry that can be thought of. The first, namely, that It is, and that it is impossible for anything not to be, is the way of conviction, for truth is its companion. The other, namely, that It is not, and that something must needs not be, - that, I tell thee, is a wholly untrustworthy path. For you cannot know what is not - that is impossible - nor utter it.[4]

Or perhaps even more directly, he says,

> I shall not let thee say nor think that it came from what is not; for it can neither be thought nor uttered that what is not is.[5]

This last sentence is crucial. It can *neither be thought nor uttered that what is not is*; we cannot correctly, rationally speak of nonbeing as having being, since nonbeing is supposed to be exactly *not* being.

This is exactly the idea we get from Parmenides's fragments and his poem *On Nature*: Being is, nonbeing is not, and nonbeing is not being. These are givens. Being and nonbeing are to be strictly separated from one another, and so we must ask ourselves what we are thinking of when we think of

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[2] Gauer et al., "Zeno of Elea," para 1.
[3] Palmer, "Parmenides," paras. 31-2.
[4] Parmenides, *Fragments*, 2.
[5] *On Nature*, VIII.7-9.

one thing *not being* such-and-such or one thing *not being* something else. But to the extent that we are thinking of these 'not being' one another, we are thinking of nonbeing. What exactly is there to be thought in nonbeing? Nothing. Nonbeing is exactly supposed to be the absence of anything to think, the negation of anything positive which could *be*, could *be* thought, and so on.

So, Parmenides maintains, whenever nonbeing or negation is involved in our discussion of 'what is', we are making an *error*, and this error is the reason that we can no longer correctly, rationally, or truthfully speak or think of anything except 'being is'. The error is that in order to think anything that involves nonbeing we must treat nonbeing as if it were being, but that is precisely the opposite of what nonbeing is. The results of this argument are profound. Anything that requires me to distinguish something from something else, specify something and rule out alternative specifications by doing so, and the like must be set aside as involving a constitutive error. *Motion, change, difference, multiplicity*, and all the rest must be cast aside as concepts based on a mistake. Motion requires that an object be in one place and then move into another which *is not* the first, at times which *are not* one another, and so on. Change requires one state transition into another state which *is not* the first. Difference and multiplicity require that one thing *is not* something else. We can see where this is going.

Rationally, we know that all these things are not real and, indeed, that it is impossible for them to be real: Nevertheless, it *seems* that motion, change, multiplicity, and the like exist. But they have to be consigned to semblances alone: Just as science has taught us that there is no intrinsic property of 'color', but there are just waves of light at different frequencies picked up by rods and cones, and so forth. Even though in ordinary experience there might *seem* to be some intrinsic quality of color objects we look at possess, this is no more than a semblance which does not actually reflect reality. In just the same way, so too has philosophical reflection taught us that there is an incoherency involved in accepting the possibility of these concepts, an incoherency which means we must consign the phenomena at issue as semblances which do not and indeed cannot reflect reality itself.

II. Zeno's Paradox: "The Dichotomy"

Zeno has many paradoxes, only one of which I discuss in this paper. These paradoxes are classed broadly as paradoxes of motion and paradoxes of plurality,[6] creating contradictions that supposedly arise from accepting the possibility of either motion or multiplicity. While the paradoxes might seem to be entirely negative on their own, we can see that they also have a positive function

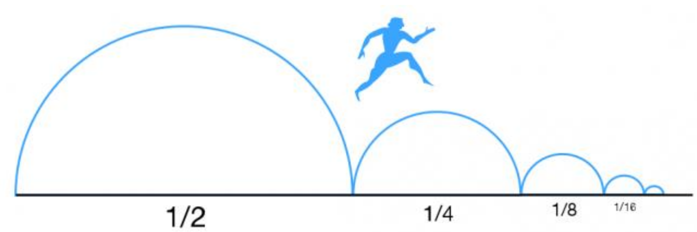
[6] Huggett, "Zeno's Paradoxes."

within Eleatic philosophy: Maybe you weren't convinced by Parmenides's original argument, but it would be strong further support for that argument if we could show that contradictions result from making the supposed 'error' he pointed out.

Zeno has several paradoxes of motion, and the most famous of these is called "the Dichotomy".[7] Often, however, due to its fame over the rest, the Dichotomy is simply referred to as 'Zeno's Paradox', which is how I will refer to it in this paper. The main source for Zeno's Paradox is Aristotle summarizing it in his *Physics*:

> The first [paradox] asserts the nonexistence of motion on the ground that that which is in locomotion must first arrive at the halfway stage before it arrives at the goal.[8]

Zeno's observation, then, is this one. For a traveler to reach the final destination, they must first go halfway there. That will take some finite amount of time, and once they reach the halfway point (without stopping, just continuously moving), they will once again have to go the remaining halfway there. Again, that takes a finite amount of time, and there is a finite amount of time left for them to travel: The traveler is not there yet. So, they must go the remaining halfway, then a remaining halfway, then a remaining halfway. What is the paradox? No matter where the traveler is during this journey, there will always be a finite amount of distance left to traverse in a finite amount of time before reaching the goal. The traveler *never reaches the endpoint*. At the moment one trip is done, another trip (however brief) must be done, then another, then another, and so on ad infinitum, never to be finished.



The paradoxical element of this is that we usually do think that motion happens and that the traveler reaches their goal. We can make this mathematically precise: Suppose, for simplicity, I am traveling at a constant rate of 1 unit of distance per 1 unit of time. Then, it will take me exactly 1 time unit to complete my journey of 1 distance unit, and so it seems that, once 1 unit of time has passed exactly, I will be at the endpoint. The trouble is, then, that Zeno's model of the same scenario gave us a reason to think we never reach the endpoint.

[7] Para. 28.
[8] Aristotle, *Physics*, 293b11.

The idea of the paradox, then, is that the presupposition that motion is possible gives us everything we need to create Zeno's Paradox. Since contradictions are always false, the contradiction the paradox constructs means that we must reject the assumption that motion is possible. This is the conclusion Zeno wants us to endorse, but of course it is not something most of us want to accept. How can we respond to Zeno's Paradox? The next four sections address precisely this question.

III. The Formal-Logical Solution

Formal logic, being a topic philosophy and mathematics share, seems a fitting place to begin my discussion of mathematical solutions to Zeno's Paradox. The Formal-Logical Solution argues that formal logic gives us a way to formulate an equivalent but stronger argument than Zeno's, one which allows us to rescue motion's possibility. This solution uses what philosophers call a 'Moorean shift', named after the 20th Century philosopher G.E. Moore.

A Moorean shift is a way to construct an argument using the propositions of another argument that it seeks to defeat. Moorean shifts defeat their target arguments by using the negation of its conclusion as a premise. They basically use the equivalence stated by contrapositive:

$$P \rightarrow Q \text{ is equivalent to } \neg Q \rightarrow \neg P.$$

So, arguments are some conjunction of propositions which entail some conclusion. Equivalent to any such argument, then, would be the negation of its conclusion used to reject at least one of the premises in the conjunction. Moorean shifts thus give us an argument as to why we do not need to accept the original argument's conclusion. The idea is that, when I am more confident that some argument's conclusion is false than I am that all its premises are true, I can supply my own argument, with equivalent validity but better soundness, to respond to it.

Let's formalize Zeno's Paradox for a concrete example of a Moorean shift. Let's say that this is Zeno's argument, phrased as a direct proof rather than a proof by contradiction: "If motion m is possible, then a traveler t reaches their journey's endpoint eventually and must complete infinitely many distinct trips to finish that journey. For any traveler j, however, if j must complete infinitely many trips to finish the journey, then j cannot reach the journey's endpoint. Hence, if motion is possible, then t reaches the journey's endpoint and does not. So, we must reject the initial assumption that motion is possible." Sentence for sentence, we could write this in formal logic as follows:

(1) $P(m) \rightarrow \big(E(t) \land M(t)\big)$

(2) $\forall j \in T, M(j) \rightarrow \neg F(j)$.

(3) $\therefore P(m) \rightarrow (E(t) \land \neg E(t))$. (Note: $t \in T$.)

(4) $\neg(E(t) \land \neg E(t))$.

(5) $\therefore \neg P(m)$.

The fourth proposition here is only up for debate if we abandon classical logic, namely, the principle of noncontradiction. So, there are really only two premises to Zeno's Paradox: the first that motion implies the setup of the paradox and the second that one dimension of this setup contradicts the other.

As might be apparent because of that, it is this second claim that is usually disputed in attempt to refute Zeno, and you might just think the fact that motion does occur implies that it is possible. So, we have reason to endorse a slightly less surprising solution with a Moorean shift:

(1) $P(m)$.

(2) $P(m) \rightarrow \big(E(t) \land M(t)\big)$

(3) $\therefore E(t) \land M(t)$.

(4) $\therefore\ \exists j \in T, M(j) \land F(j)$.

(5) $\therefore \neg\big(\forall j \in T, M(j) \rightarrow \neg F(j)\big)$.

In English, this argument reads as follows: "Motion m is possible. If motion is possible, then a traveler t reaches the journey's endpoint and must make infinitely many distinct trips in the journey to finish it. Therefore, t reaches the journey's endpoint and must make infinitely many distinct trips in the journey to finish it. Hence, there exists some traveler j $(= t)$ who reaches their journey's endpoint and must make infinitely many trips in the journey to finish it. In other words, it is not the case that, for any travelers j, if j must make infinitely many trips to complete their journey, then the traveler does not reach the endpoint." This is, of course, the denial of one of the assumptions of the paradox.

Even though this Formal-Logical Solution does give us some *reason* to reject the conclusion of the paradox, there are difficulties here. Zeno does not just state the claim the Moorean shift concludes is false as given; he gives us a *reason not to believe it*, namely his argument about how a point never occurs in the divided-up journey where the traveler can be said to have 'finished' the journey (reached its endpoint) with no time remaining. The Formal-Logical Solution does nothing to diffuse the force of his construction. Beyond that, Moorean shifts are not generally persuasive to the opponent. At best, they allow us to be convinced that we are not being irrational because we are ignoring an argument, but they will not persuade someone who is clearly ready to give up the thing we find to be so certain.

For these two reasons, we must look to solutions which take Zeno's construction itself more seriously. These are where the more quantitative mathematical solutions come onto the scene. I will begin with what I call the *Finite Bounds Solution*, as it will lead us into the Infinite Sum Solution.

IV. The Finite Bounds Solution

If we are going to find a more satisfying answer to Zeno, we will have to challenge his argument that a sequence of infinitely many journeys can never reach its endpoint. One interpretation of this argument is that the traveler never reaches the endpoint because the journey *takes an infinitely long amount of time*. This, I should note, is not entailed by the version of the argument I presented above, though it is a common interpretation of Zeno's argument. Aristotle himself interprets Zeno this way.[9]

I begin, then, by presenting a modernized version of one of Aristotle's own solutions, which I call the *Finite Bounds Solution*. The basic observation that Aristotle makes is this one: Whenever we divide the distance traveled in the journey, we must also divide the time the journey takes.[10] Just as the distance that we have left gets infinitely small as we continue to make infinitely many divisions within the one unit of distance, so too will the time it takes to traverse that unit become infinitely small. Aristotle thus distinguishes two kinds of infinity[11] he thinks Zeno's Paradox conflates:

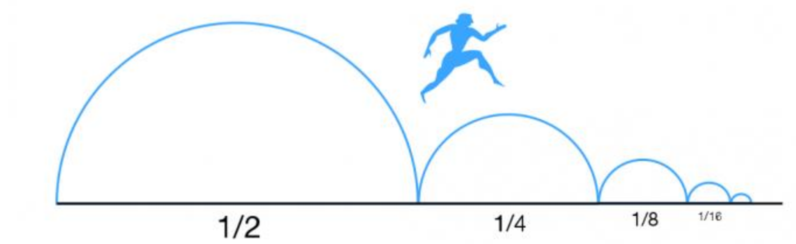
(A) *Infinite in Divisibility*: Some things are called infinite in the sense that they are infinitely divisible.
(B) *Infinite in Extremity*: Some things are called infinite in the sense that they extend infinitely far.

Aristotle thinks that Zeno is conflating the first kind of infinity with the second. The time required to complete the journey will be finite, and we can surely divide this interval of time into as many divisions as we like. We assign a rule for dividing, like Zeno does, which in principle never can be listed in its entirety. But it nevertheless remains true that we are just infinitely *dividing* an interval which is not infinite *in extremity*, an interval with *finite bounds*.

The force of this solution is more easily seen by thinking about the problem geometrically. Recall that to make things simpler we said that our traveler will be going one unit distance per one

[9] Aristotle, *Physics*, 233a22.
[10] Ibid. 233a13-24.
[11] Ibid. 219b1-2

unit of time, so that it takes x unit(s) of time to travel x unit(s) of distance. This direct translation between distance and time passed allows us to reinterpret the image from earlier. We can think of the distance line as also a timeline, so that the divisions made in distance are equally divisions in time. Going one half the way means using one half the time, and so on. That in mind, observe again:



Nobody is confused that the total distance that the runner is traveling is contained within the interval $[0,1]$, since the journey takes one unit of distance overall. We can slice this interval in half no matter how many times we would like, but the total distance we will need to traverse in the journey will never get over 1 unit of distance. Thus, the time passed and the distance passed will not go on to infinity; they are rather contained in a finite interval, and therefore the journey cannot take an infinite amount of time. Thus, it seems, the traveler will reach its endpoint in a finite amount of time (viz., 1 time unit).

In modern analysis, we can further formalize this idea using the idea of *upper and lower bounds*, or a *bounded* sequence. Let me define these terms:[12]

- **Definition 1**: An *upper bound* for a set $A \subseteq \mathbb{R}$ is some real number $u \in \mathbb{R}$ with the property that, for all $a \in A, a \leq u$.
- **Definition 2**: A *lower bound* for a set $A \subseteq \mathbb{R}$ is some real number $l \in \mathbb{R}$ with the property that, for all $a \in A, a \geq l$.
- **Definition 3**: A sequence $a_1, a_2, \dots, a_k, \dots = (a_n)$ is *bounded* if there exists a positive real number $M \in \mathbb{R}^+$ such that, for all natural numbers $n \in \mathbb{N}$, $|a_n| \leq M$.

The Finite Bounds Solution could be stated formally in two ways. The general way of stating it is that that, if the time or distance elapsed is contained in an interval with finite length, then the time required to reach the journey's endpoint cannot have an infinite length. I will not discuss this in depth, but we

[12] Abbott, *Understanding Analysis*, 15, 49.

can even prove here that, since Zeno accepts the sequence is monotonically increasing, such a sequence contained in $[0,1]$ the must converge, by the Monotone Convergence Theorem.[13]

I will restrict myself to the argument that an infinite length of time cannot be contained in an interval of time with finite length. I will rigorously prove that the set of amounts of time required to complete the trips in the journey is in fact bounded by two finite values. However, I will need a general expression for how much time will be elapsed in total after the nth trip in the journey is complete. Just looking at some examples, it is not too hard to come up with this general formula:

- $t_1 = {}^{1}/_{2}$.
- $t_2 = {}^{1}/_{2} + {}^{1}/_{4} = {}^{3}/_{4}$.
- $t_3 = {}^{1}/_{2} + {}^{1}/_{4} + {}^{1}/_{8} = {}^{7}/_{8}$.

  … Etcetera

The time required to finish the first trip is one half a time unit. The time required to complete the second is one half a time unit, plus one half the remaining half, or one fourth. And so on. It seems the total amount of time it will take during these trips is always going to be some power of 2 in the denominator with one minus that denominator in the numerator. That is, all our examples fit the form

$$t_n = {}^{(2^n - 1)}/_{2^n} = 1 - {}^{1}/_{2^n},$$

and it is not too hard to prove this using mathematical induction. In mathematical induction, we prove that a fact is true for some base case, and then we prove that if it is true for one case, then it is true for the next case. This allows us to infer from the base case that a second case is true from the first, then a third from the second, then a fourth from the third, and so on, so that we can say any case of the statement is true. Here, we see that

$$t_1 = {}^{(2^1 - 1)}/_{2^1} = {}^{1}/_{2},$$

as predicted. Now, for any arbitrary natural number $k \in \mathbb{N}$ (which includes 1), suppose that t_k has the desired form. Then, we will have to add the remaining half of the time or distance left after t_k to obtain t_{k+1}, which gives us

[13] Ibid. 56.

$$t_{k+1} = \frac{2^k - 1}{2^k} + \left(\frac{1}{2}\right)\left(1 - \frac{2^k - 1}{2^k}\right) = \frac{2^k - 1}{2^k} + \left(\frac{1}{2}\right)\left(\frac{1}{2^k}\right) = \frac{2^k - 1}{2^k} + \left(\frac{1}{2^{k+1}}\right)$$

$$= \frac{2(2^k - 1)}{2(2^k)} + \left(\frac{1}{2^{k+1}}\right) = \frac{2^{k+1} - 2 + 1}{2^{k+1}} = \frac{2^{k+1} - 1}{2^{k+1}},$$

which is precisely the formula we were trying to prove t_{k+1} had. So, if t_k has the form desired, so does t_{k+1}, which means that we can infer that any $n \in \mathbb{N}$ yields a t_n with the form I proposed above.

This puts us in the position to prove that the set of all times elapsed upon the completion of any trip in the journey, which I will denote $T = \left\{ (2^n - 1)/2^n : n \in \mathbb{N} \right\}$, is bounded above by 1 and below by 0. Since the numerators and denominators of every element in this set are positive, any element in it will be positive, that is, greater than zero. Hence, 0 is a *lower bound* for T, or T is *bounded below* by 0. Similarly, any element in the set can be expressed as $1 - 1/2^n < 1$ for any $n \in \mathbb{N}$, since $1/2^n > 0$. Therefore, 1 is an *upper bound* for T, or T is *bounded above* by 1. This implies that every element in T is contained in the closed interval $[0,1]$ (i.e., $T \subseteq [0,1]$), and the length of this interval is 1. Accordingly, no matter how long we take in completing trip after trip – let it go infinitely long – the journey will neither take nor require an infinite length of time; it will never be able to get over 1. This same result implies that every t_n in our sequence will have a magnitude $|t_n|$ such that $|t_n| \leq 1$, which means the sequence is *bounded*.

As this solution stands, however, it cannot be regarded as entirely satisfying. As I mentioned, it could be rigorously shown in this way that the sequence converges to the endpoint of the journey, but this solution does not explicitly show the traveler's journey sends them to the endpoint. It just argues against the infinite length of time this supposed to take. Consequently, I will turn to a different proposed solution in which, I think, we can find a similar, more satisfying solution in the short space of this paper: one which does show the series converges. This is what I call the *Infinite Sum Solution*.

V. The Infinite Sum Solution

What we need is an argument for how completing these infinitely many trips takes the traveler to some finite value (hopefully the endpoint 1). Fortunately, the series (infinite sum) in Zeno's Paradox is one of the most famous and commonly discussed series, usually covered in introductory calculus courses. It is an example of a *geometric series*, which in general is defined to have the form

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 \dots$$

It can be proven in general that these series converge to a finite value whenever $|r| < 1$. The value to which they converge is even by the general formula

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}.$$

Everything works out here wonderfully. Represented this way, the sum of all the times or distances passed would be

$$\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^3 \dots = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^n = \frac{\left(1/2\right)}{1-\left(1/2\right)} = 1,$$

exactly the value we wanted this sum to be equal to. It seems then, that the traveler *does* complete infinitely many trips and end up exactly where we would hope: at the end goal.

I will not prove all this general statement, but it is not too involved to prove it just for our particular case of summing the halfway time slices. Of course, just leaving a … in the expression of a sum equal to some finite value is not mathematically rigorous. The precise definition at issue is the convergence of an infinite series or infinite sum:

- **Definition 4**: An infinite series *converges* to some value $a \in \mathbb{R}$ if the sequence of partial sums converges to $a \in \mathbb{R}$.[14]

The convergence of an infinite series, then, is defined in terms of sequences. It suffices here to think about sequences as infinitely long lists of terms $a_1, a_2, \dots, a_k, \dots = (a_n)$, though this is not a precise definition. Convergence is defined by a specific type of sequence, that of partial sums. The sequence of partial sums is a familiar one, defined so that each nth term of the sequence of partial sums is

$$s_n = a_1 + a_2 + \cdots a_n = \sum_{i=1}^{n} a_i.$$

[14] Abbott, *Understanding Analysis*, 57.

Fortunately, we already calculated and proved a general formula for the infinite sums in the case of Zeno's Paradox. Recall that

- $t_1 = {}^1\!/_2$.
- $t_2 = {}^1\!/_2 + {}^1\!/_4 = {}^3\!/_4$.
- $t_3 = {}^1\!/_2 + {}^1\!/_4 + {}^1\!/_8 = {}^7\!/_8$.

  … Etcetera

So, in our case, the sequence we need to converge to 1 is the sequence (t_n). But in general, to understand what it means for an infinite series like this to converge, we will have to understand what it means for a sequence to converge:

- **Definition 5**: A sequence (a_n) *converges* to some value $a \in \mathbb{R}$ if, for any positive real number $\varepsilon \in \mathbb{R}^+$, there exists a natural number $N \in \mathbb{N}$ such that, for any natural number $n \in \mathbb{N}$ with $n \geq N$, $|a_n - a| < \varepsilon$.[15]

In effect, this definition gives a mathematically rigorous way to say that the terms in this sequence get 'infinitely close' to the value to which they converge a. It says, if you give me any positive real number, I can find a point in the sequence where that term and every term after it differ from a by less than the arbitrary real number you gave me. The sequence 'approaches' this value, sometimes called the *limit* of the sequence, 'infinitely closely'. Using this definition, we can prove our result:

Result: $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^n = 1.$

Proof. To prove this, we simply need to show that (t_n) converges to 1. Take any positive real number $\varepsilon \in \mathbb{R}^+$, as small as you like. We need to find a number which indexes a point in the sequence at which, from then on, the distance from 1 is smaller than this arbitrary number. I claim that any natural number $N \in \mathbb{N}$ satisfying ${}^1\!/_{2^N} < \varepsilon$ will work for this. Let us show this. Take any $n \in \mathbb{N}$ for which $n \geq N$. Then, the difference of the nth term from the proposed limit of the sequence is

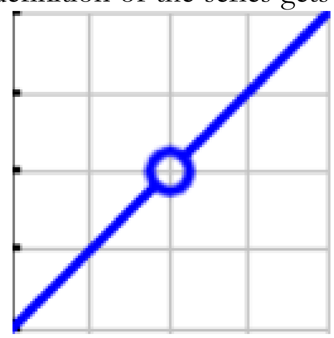
$$|t_n - 1| = \left|\left(1 - {}^1\!/_{2^n}\right) - 1\right| = \left|{}^1\!/_{2^n}\right| = {}^1\!/_{2^n} \leq {}^1\!/_{2^N} < \varepsilon,$$

which means any such nth term will differ from 1 by less than the given arbitrary value. In other words, the sequence gets infinitely close to 1, or it converges to 1.

[15] Ibid. 43.

Of course, this is exactly the solution we would have hoped for. It seems that we sum up all the trips in the journey that Zeno wants, and we obtain the length of time we would expect the overall journey to take: 1 unit of time. Likewise, this result implies infinitely many trips' distances add up to the total distance we wanted to travel: 1 unit of distance.

There are two problems here, however. The first is a mathematical problem, and this problem will lead me to a philosophical-interpretive one. This solution will only work if we assume that the traveler's motion is continuous, so that the fact that the sequence in the definition of the series gets 'infinitely close' to 1 entails *that it actually reaches* 1. Consider the graph to the right. Let's say that the x-axis is distance traversed by some traveler, and the y-axis is the time elapsed while traveling that distance. The line described here is the function



$$f : (\mathbb{R} - \{2\}) \to \mathbb{R} \text{ defined } f(x) = x.$$

We can divide up our traveler's journey here in a very similar way to Zeno. Define the sequence (b_n) as follows

$$b_n = f\left(2 - {}^{1}/_{n}\right),$$

so that the terms in the sequence go $f(1), f\left({}^{3}/_{2}\right)$, ..., etc. As is clear on the graph, this sequence is going to converge to 2, *but this convergence does not entail that the traveler reaches the end goal* at a distance of 2 and a time elapsed of 2. This is important because, as we saw, to say an infinite series converges is just to say some kind of sequence converges, and so the Infinite Sums Solution will not on its own be able to ensure that the traveler reaches their endpoint just by pointing to a the limit of a sequence of partial sums. The traveler could get infinitely close to this value without ever arriving there, *as Zeno's Paradox originally suggested.*

This brings me to the philosophical or interpretive issue this raises: Namely, I do *not* think this 'infinite amount of time' interpretation of Zeno is the strongest or most charitable interpretation. As I presented it earlier, Zeno's Paradox *only* required that the traveler would never reach their endpoint. It never said this 'never' means 'forever', that is, an infinite amount of time. In my view, the strongest interpretation of Zeno would interpret Zeno's Paradox as saying that the traveler can traverse $[0,1)$ in a finite amount of time, but his argument suggests that they will still not reach the endpoint 1.

VI. The Modeling Solution

That problem with the Infinite Sum Solution will also defeat the Finite Bounds Solution, if we accept my interpretation that Zeno does not presuppose the traveler's journey will take an infinite amount of time. Indeed, his paradox's construction gives us a reason to think the traveler does not reach the endpoint, which means a reason to think motion would not be continuous (as the Infinite Sum Solution tacitly presupposes).

So, convergence alone will not make for a satisfying mathematical solution to Zeno's Paradox. But this failure also suggests a new way to try to solve the paradox mathematically: If Zeno is denying or concluding that motion is not continuous, *maybe he is just modeling motion incorrectly*. This kind of solution to the paradox is what I call the Modeling Solution, and Aristotle provides a solution of this type himself, which I will examine here.

Aristotle criticizes Zeno for modeling motion as fundamentally discrete, while he ought to have modeled it as fundamentally continuous. This criticism is tied to Aristotle's first one, examined in the Finite Bounds Solution. Aristotle distinguished the two kinds of infinity because he thinks motion and time are *continuous*, and he defines continuity as follows:

- **Definition 6:** "By continuous I mean that which is divisible into divisibles that are always divisible."[16]

I call this *Aristotelian continuity* to distinguish it from the modern mathematical concept of continuity I will discuss later. So, the interval $[0,1]$ is Aristotelian continuous because any division we make to split it into two will result in two new intervals that themselves can always be divided again. But the set $\{1,2,3\}$ is not Aristotelian continuous because it can be divided into the sets $\{1\}$ and $\{2,3\}$, the first of which is not divisible any further.

This suggests what Aristotle's complaint with Zeno will be.[17] Zeno treats *one and the same motion* as if it were a discrete sequence of *distinct motions*. Really, there is only one process of motion happening, and this motion is Aristotelian continuous, which means it potentially contains all of the divisions that Zeno wants it to contain. If, Aristotle maintains, the motion in some way ceased after each interval, Zeno would be right that the journey could never be completed. But really the motion is not ending

[16] Aristotle, *Physics*, 232b24-25.
[17] Ibid. 263a4-263b8.

and starting; it is just continuing as one and the same motion, and time passes with it. In effect, Zeno would be correct if it were *modeling* a sequence of distinct motions, but instead he is incorrectly modeling one and the same continuous motion as a sequence of distinct motions in the plural. Aristotle distinguishes these two by saying that the way Zeno represents things treats them as if the divisions were *actual*, whereas in Aristotelian continuous motion they are merely *potentially* divided.

I doubt that Aristotle's solution, as currently stated, would solve Zeno's Paradox without some modifications. For example, it seems to me that the interval $[0,1)$ is Aristotelian continuous and potentially contains all trips in the journey, and this is the interval that Zeno is arguing our motion can successfully traverse in time and distance. After all, no matter where I split the interval, the result will be two intervals which themselves can be divided just as easily as the original.

Nevertheless, the basic insight of Aristotle's solution stands. Whether it is the concept of motion or motion in nature that we are trying to represent using these mathematical modes, in either case we are trying to *represent* something mathematically, and the model we use to represent it can be *incorrect*, as Aristotle thinks Zeno's is. For the rest of this section, I will sketch a solution of this kind which, I think, would successfully avoid Zeno's Paradox by arguing he models motion incorrectly.

In order to do this, I need to specify what I make some observations about the concept of motion or about motion in nature, whichever we think the paradox deals with. I will stipulate that the following things are true about motion:

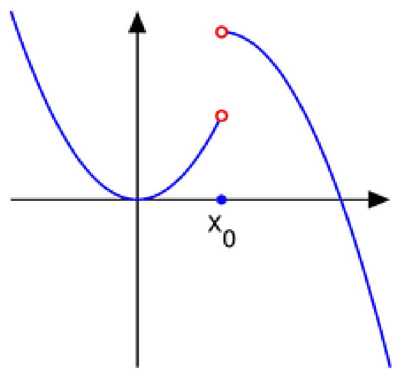
(1) Motion is a position function, that is, a rule assigning a position in space to physical object for any given point in time.
(2) Physical objects are spatiotemporal, that is, there is never a point in time or space where they exist that they do not have a position in space and time.
(3) Motion is continuous in the modern mathematical sense, which I will explain informally in a moment.

In modern mathematical terms, (1) gives us a way to represent motion mathematically. (2) guarantees that motion will be defined for any time; I assume there are no 'gaps' in time, points where time does not exist on a timeline, so that the position function is defined for the whole real line. Using these two axioms of motion, I might suggest the following function to describe our traveler's journey:

$$s(t) = t \text{ if } t \in [0,1] \text{ and } s(t) = k(t) \text{ if } t \in (\mathbb{R} - [0,1]),$$

where $k(t)$ is just an arbitrary continuous function which can be continuously pieced together with the $s(t)$ over $[0,1]$, so that we will let our traveler have some freedom of movement. But what does this continuity involve? What does (3) mean?

I do not think the rigorous definition of continuity is helpful enough to try to fit it into the limited space of this paper, and so I will not provide a rigorous definition of continuity here. It suffices to think of a continuous function graphically as a function without gaps or jumps; these functions are curves which, if you want to think of it this way, could be drawn without lifting your pencil. So, along with the line with a hole we saw already, the following is also not a continuous function:



So, when we stipulate that the function is continuous over the real line, even if Zeno insists that the motion does not reach its goal *in the journey* we are considering, we would have to represent it as

$$s(t) = t \text{ if } t \in [0,1) \text{ and } s(t) = k(t) \text{ if } t \in \big(\mathbb{R} - [0,1)\big).$$

But because the function for this motion is defined over the real line and continuous, there cannot be a gap at $x = 1$, which means that there must be some value $s(1)$. If this value were anything other than $s(1) = 1$, there would have to be some jump between the approach to 1 which gets infinitely close to it and $s(1)$ itself. That is, the function could not be continuous, violating (3). The result, then, is that this model of motion does demonstrate that the traveler reaches the endpoint 1 at time 1.

Indeed, we also get a version of Aristotle's claim that the journey 'potentially' contains every journey Zeno claims we need to make. We can show this by observing that every endpoint of a trip in the journey is contained in the set of all points traversed. That is,

$$\left\{{}^{(2^n - 1)}\!/_{2^n} : n \in \mathbb{N}\right\} \subseteq f([0,1]) = [0,1].$$

So, not only does Zeno's traveler reach the endpoint, but they also perform all the trips Zeno demands that they perform. All these trips are contained in a single continuous motion, just as Aristotle claimed. It seems, then, that this model gives Zeno everything that he wanted and yet also provides an explanation as to why the runner would reach the endpoint: simply a better model.

I will leave it to the reader to judge whether this kind of reply is adequate or not. I think it is the most satisfying one we have seen, but Zeno might remain unconvinced. After all, his argument gave us a reason to think these axioms about motion I had to introduce in order to construct this solution *might not be true at all*. It seemed that when we divided up the motion into trips contained in the journey, no matter how many trips we complete, there will always be another lingering trip which will take a finite amount of time and must be completed before we reach the end goal. The whole trip is plagued by a 'not yet' which, it seems, allows us to *approach* the endpoint without ever being able to have finally *reached* it. Why is this understanding of the movement flawed? Axioms give us principles to avoid the conclusions it suggests, but the question at issue is which axioms we should accept.

VII. Concluding Reflections on the Paradox

What I think the Modeling Solution elucidates, however, is something that the previous two mathematical solutions do not. Zeno is fundamentally attacking the concept of continuous motion from A to B by showing how that motion can be represented *in a valid way* which leads one to conclude that it never will reach an arbitrary, given end goal. Now, it is entirely possible that the way Zeno is trying to reconstruct the journey is mathematically inconsistent, but I think what we have seen already suggests that, on its strongest interpretation, it is not. If we want to challenge Zeno, then, perhaps the best way is by challenging he models or represents motion, rather than trying to find a mathematical error in his thinking somewhere. The insight of the Modeling Solution is to challenge the claim that Zeno *is* representing motion 'in a valid way'.

Maybe there is a mathematical problem in Zeno's Paradox, on the stronger interpretation I have suggested here. But I am doubtful that there is, and the solutions we have examined – except the Modeling Solution – do not seem to work. Perhaps, then, there is no *strictly* mathematical solution to Zeno's Paradox. The Modeling Solution blends the kind of thinking done in philosophy and mathematics; especially if you hold Zeno is primarily attacking the *category* of motion in nature per se, the Modeling Solution seeks to find a way to determine what is true *in the first place* (before axioms are presupposed), what motion even *means*, and the like. Once those things are determined and connected to mathematics, mathematics gives us precise ways to flesh out solutions concretely and confirm that they do indeed avoid the problem. I find it rather fitting, then, that in creating this dialogue between philosophy and mathematics, the conclusion I am led to suggest because of it is that philosophy and mathematics not only *are* connected but, in some areas, *must* be connected.

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