FROM MATHEMATICS TO ORIGAMI AND BACK

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1. Introduction

Origami, also known as paper folding, is the subject studying how to fold a piece of flat paper (often a rectangle) into creative or lifelike motifs. In this paper, I would like to discuss how mathematics intervenes modern origami. The connection between these two subjects not only allowed math to shape origami from a childhood toy into an art form, but, in turn, also let origami inspires new math studies as well as applications in other scientific fields.

2. History of Origami and the Role Math Played

Mathematics' impact is strongly reflected on the history of origami. When math is introduced into the design process of it in 1900s, we see the complexity of the artworks increased exponentially in that century. With only a hundred year's time, the subject of origami grew in scale and significance, became a serious art form that worth studying. The sudden flourishment is achieved by math in general.

2.1. **Early history of origami.** Origami did not origin from a particular site or time. Since paper was invented by Chinese in 105 CE and the word origami came from Japanese "ori (fold)" and "kami (paper)" with solid evidence showing origami existed in 1600s, we view China and Japan as where origami started from [12]. During the years between 1600s and 1900s, origami functioned as childhood toy or for ritual uses, and exploration was the main method to create new designs. It was until the twentieth century did deliberate planning take place and people heavily pursue complicated figures and aesthetic meanings.

2.2. **Encounter of origami and math in the twentieth century.** Mathematics was first connected to origami in the nineteenth century, when T. Sundara Row published "Geometric exercises in paper folding" in 1983, showing how to construct geometric shapes by origami in replacement of the traditional straightedge and compass constructions. In the introduction of his book, Row discussed his approaches to folding geometric figures and proofs as "These exercises do not require mathematical instruments [...] In paper-folding several important geometric processes can be effected much more easily than with a pair of compasses and ruler, the only instruments the use of which is sanctioned in Euclidean geometry."[13] His book, originally inspired by the kindergarten gift he received years earlier, drew the eyes of other

Date: March 16, 2022.

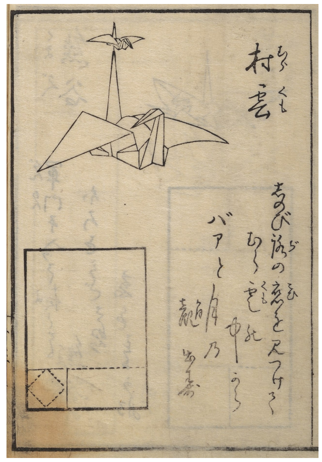

FIGURE 1. Shimokobe Shusui, The folding of two origami cranes linked together. 1797. [14]

mathematicians to participate in the study of origami. However, before mathematicians made new conclusions on their own field with paper folding as a new tool, they first reformed the world of origami.

The change of origami from a casual toy into a serious art form happened in the twentieth century. It was led by several artists and mathematicians with their innovation on the strategy of designing the works. One of the pioneer is Akira Yoshizawa, who created the common language to describe the design in 1954. His effort standardized the transcription of origami works, which made the duplication of certain artworks achievable [5] [11]. He used two kinds of dotted lines to represent the only two types of crease line: mountain fold and valley fold. Mountain fold represents any crease folding downward, creating a mountain-like shape, and valley fold means the opposite. Such representations allow origamists to communicate efficiently and effectively with sketches, cheaper and easier than by the photos of actual papers.

Another preeminent origamist is Jun Maekawa, a mathematician who started the first wave of origami revolution by using conscious design decisions instead of random attempts in 1986. The complicated facial expressions and the fingers of his most inspiring work, "Demon," were the evidence of mathematical thinking interference. The crease lines also revealed the thorough planning of the entire paper before the folding process. Since then, math took over origami's design of overall structures [3]. The devising of the paper was later fully taken over by mathematical algorithms in 1990s when Robert Lang published his computer program called *TreeMaker* [11].

Besides the general planning of paper and the structure created, the mimicry of the texture of objects in origami works also built on mathematical studies. The second wave of origami revolution, led by Satoshi Kamiya, incorporated the idea of tessellation and created the new concept of crease patterning, known as origami tessellation [3]. The direct impact by Kamiya is that new studies focusing specifically on origami tessellation developed. Another inspiration is the idea to implement more mathematics models into origami, in another word, to build on the past experts [10].

Figure 2. Origami "Devil" by Jun Maekawa [2]

Origami transformed from a craft to an art form. By the end of the twentieth century, mathematics used in computational origami actually became indispensable in this subject, as the design process for the desired details of origami was too computationally complicated [5]. Before we discuss how such result was accomplished, we fist need to define origami properly so that it can be quantified in programming languages.

3. Modern Definition of Origami

There are many branches of modern origami: modular, wet folding, crumpled, and so on. To define origami in modern terms, we have to classify it by the technique it used, because each type of origami is very distinct. Though there are more to explore with 3-dimensional origami and the ones with curved creases, to keep this paper in reasonable length and organized, we mainly discuss Euclid geometry, so we only talk about one and the most popular branch of paper folding: flat origami.

Flat origami generally requires only one piece of square or rectangular shaped paper, and construction process with no cuts, folding straight lines only. An additional limitation for flat origami is that the finished result must be able to "be pressed in a book without crumpling or adding new lines" [4]. To put this into more mathematical words could be hard, because we cannot simply claim that each fold in flat origami must be 180°(For example, the cranes). There are many scenarios that the final model appears with volume, and the 180°is only valid with its base.

The origami artworks below are all in the class of flat origami. They have various shapes and styles, but they fall in the restrictions of flat origami. To study them, especially from the mathematical point of view, we often focus on their "blueprint," that is, the crease lines on the flat paper with Yoshizawa's notations. How significant is math to origami? In the next section, we will discuss the design process of origami works, and the crucial part of mathematics would be revealed.

I would also like to explain a few terms used in origami:
Mountain fold and **Valley fold** are the (only) two kinds of folds in origami, usually

denoted by two different types of dotted lines. These lines are jointly called *crease lines*.
Crease pattern is the layout of all the crease lines of an origami on an unfolded paper, the blueprint of it.
Crease assignment refers to the action of assigning an unclassified crease line to mountain or valley.
A Base of an origami artwork is a projection of the actual model onto a flat surface, with each furcation as a separate flap.

4. Quantify Origami for Computer Simulation

4.1. **Modern approaches: computational origami.** As modern sophisticated design of origami requires a well-planned crease pattern before the actual folding. As origamists tried to incorporate computer programming into the computational complicated calculation process, they first decided to study the crease pattern of the folded works and establish the rules that the program has to follow, so that the patterns created are feasible.

There are four rules that are frequently mentioned by nowadays origamists among their attempts:

- If a crease pattern is flat-foldable, the crease pattern must be double-colorable.
- For the crease pattern of any flat origami, the difference between the mountain folds and valley folds emanating from a single vertex interior of the paper is always two. (Maekawa's Theorem)
- A collection of creases meeting at a vertex are flat-foldable if and only if the sum of the alternate angles around the vertex is π. (Kawasaki's Theorem)[4]
- A sheet can never penetrate a fold.

Are the rules above sufficient to create a crease pattern? Kawasaki's Theorem succeeds in a single-vertex pattern, but the "only if" part failed the sufficiency when there are multiple vertices on a crease pattern [1]. What is somehow disappointing is that, after origamists studied "what rules does the crease pattern has," they failed to distinguish whether a given crease pattern is flat-foldable in a timely manner. In 1996, Bern and Hayes published a proof claiming the determination of a given crease pattern is flat-foldable is NP-complete [1]. The strategy to create large amount of random crease patterns by computer then to justify and select the foldable ones was stuck. The idea of creating a successful computer designer became questionable. However, origamists did not stop their attempts to make computer their helper on the complicated design process. From the newly discovered design technique, circle-river packing, Lang developed the computer program *TreeMaker*, which can help designers to create the crease pattern of the base.

4.2. **Mechanism behind TreeMaker.** *TreeMaker* is a computer program invented by origamist Robert Lang in the 1900s. His main goal of this program is to explore the mathematical theory of how to design a base out of his academical curiosity. As he undated his program with faster and stronger algorithms and functions, *TreeMaker*

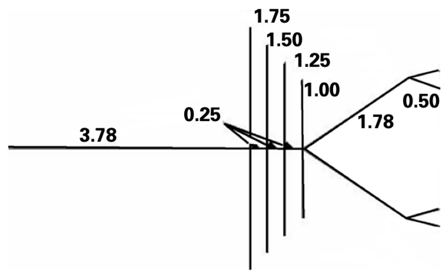

FIGURE 3. The stick figure of "Scorpion" by Robert Lang. [3]

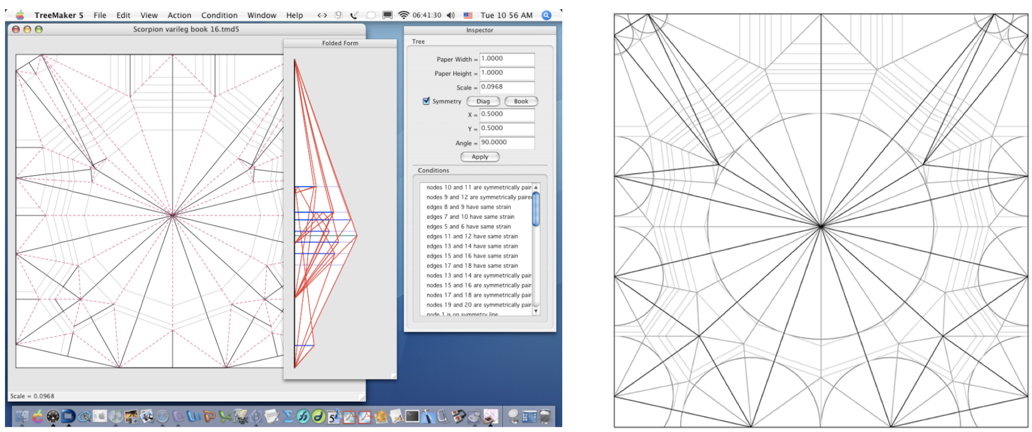

FIGURE 4. The crease pattern of "Scorpion" by *TreeMaker* [11]

became a powerful tool for the construction of the crease pattern for all kinds of origami bases [11].

The circle-river packing technique is inspired by the concept of trees. Taking the origami bases into general shapes, the tail of a cat or a finger of a demon can be represented as a "branch." If we think about this in graph theory, to get a nice and sharp edge of length a with leaf vertex on the paper, we need to astringe the circular area with radius a around that vertex together. In another word, we need a circle (or a fraction of circle, if the vertex is close to the boundary of the paper) to create each branch. From this approach, we change the problem of folding a tree from a piece of paper to packing the circles that has radius of the length of the branches on that paper [3]. *TreeMaker* aims to find the most efficient way to solve the circle packing problem, that is, find the smallest paper to pack all of the circles. An example by Lang is shown below. To create a scorpion, the user input the stick figure of the design with expected length of each branch labeled. *TreeMaker* would output the crease pattern for the base, and by adding more details to the folded base, the designer can get a satisfying model.

TreeMaker helps the designing process because it can solve the origami bases that are much more than what a person can solve by hand. What Lang viewed his

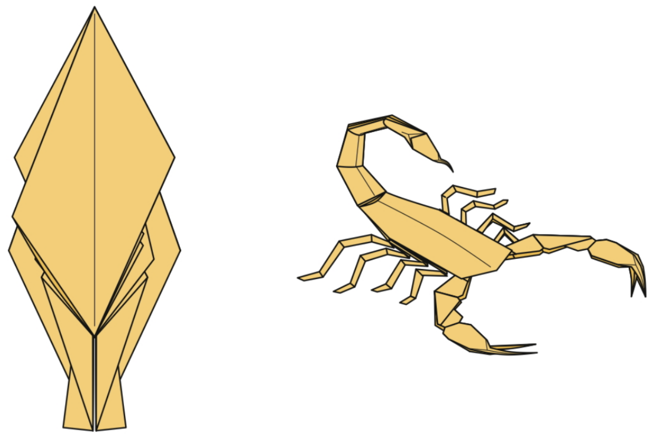

FIGURE 5. The base and actual adjustments of "Scorpion" by Robert Lang [11]

program impressing is the combination of novelty with efficiency *TreeMaker* achieved [11]. Given a stick figure, *TreeMaker* often provides the most efficient way of using the paper, which sometimes turns out to be an entirely new structure in the origami world. That being said, this computer program has some kind of "creativity," helping the innovations of origami as a whole.

5. Importance of Origami on Mathematics

Not all mathematicians care about how an art form is significantly supported by the knowledge of math, but they do get excited if a system with new axioms is created for mathematical proofs. In mainly the field of Euclid geometry, origami inspired new models with its unique axioms, much stronger than the straightedge and compass system.

I like to think about origami as a variation of straight-edge and compass, a more advanced version: It is the boundary that acts as straight edge and T-square that makes it powerful. Another difference in my observation is the fact that a compass allows us to create a specific length with one fixed endpoint. However, paper-folding allows us to move around the plane with both endpoints flexible. The folded part also functions as a record of the distance and angle that the point and edge we moved.

The study of origami geometric constructions was led by the seven "axioms" that are brought up starting from 1990s. At the First International Meeting of Origami Science and Technology in 1989, Japanese mathematicians Humiaki Huzita and Benedetto Scimemi published six distinct ways to create a crease by aligning combinations points and lines, known as the Huzita axioms:

(1) Given two points p_1 and p_2, we can fold a line connecting them.
(2) Given two points p_1 and p_2, we can fold p_1 onto p_2.
(3) Given two lines l_1 and l_2, we can fold line l_1 onto l_2.
(4) Given a point p_1 and a line l_1, we can make a fold perpendicular to l_1 passing through the point p_1.

(5) Given two points p_1 and p_2 and a line l_1, we can make a fold that places p_1 onto l_1 and passes through the point p_2.
(6) Given two points p_1 and p_2 and two lines l_1 and l_2, we can make a fold that places p_1 onto l_1 and place p_2 onto l_2. [6]

Not until 2002 did the seventh axiom is discovered by Koshiro Hatori:

Given a point p_1 and two lines l_1 and l_2, we can make a fold perpendicular to l_2 that places p_1 onto l_1.

Another story is that the French mathematician Jacques Justin published a paper in 1989 with all of the seven axioms. Hence, these axioms are also called Huzita-Justin axioms [6].

Proved by Lang in the 2000s, these axioms are the only unique folds that one is able to make. Just like the story of Euclidean geometry, the axioms may not be flawless in larger context, as further efforts into this study are needed. However, from the starting point of Sundara Row and his observation that paper folding results in stronger axioms than straightedge and compass, other mathematicians built on his idea with acceptance of the axioms, resulting in astonishing outcomes. One of the most accomplished result is by Robert Lang. Beside his *TreeMaker* which focuses on the artistic aspect of origami, he also created a program called *Reference Finder* which has more mathematical significance. By inputting the coordinates of the position of a point on a one-by-one square and a single click, this on-line program can show the steps to find that point by just folding the paper, with error less than 0.002 [8]. This program Incorporated all of the seven Huzita-Justin Axioms [9]. He also wrote an article "Origami and Geometric Constructions" specifically focusing on how to divide a unit-squared paper into n equal distanced folds [7].

According to Lang, geometric concepts are used in *TreeMaker* and some additional algebra is used in *Reference Finder*. There are many discussions inspired by origami in the mathematics field, which can be found here. `http://mars.wne.edu/ thull/origamimath.html`

6. Origami's Application in Other Scientific Disciplines

The new structures discovered by *TreeMaker* are not significant to origami artists. In fact, the structures of origami are sometimes used by engineers in their own fields. For instance, the space projects had utilized certain structures created by flat origami to pact their large piece equipment into small cases, which can be fully expanded or folded again with a simple, small-scaled action that requires little energy. [10]

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