A brief history of the continuum hypothesis

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Abstract

Cantor's insistence on his Continuum Hypothesis, and the interest in it of many others largely fueled the creation of set theory. We look at how the Continuum Hypothesis developed, defining and exploring the mathematics around its creation, and the math developed in its wake. The end result of this particular journey is a possibly unsatisfying independence, but the creations in the pursuit of this are essentially foundational to higher math today.

1. Introduction

"Naive set theory," an idea of being able to have collections of mathematical objects without rigorously establishing any axioms that can be used, has been employed essentially since the beginnings of mathematics. Aristotle (384–322 BCE) is an early example of someone using similar ideas to those that we now use regularly in set theory, in his work on logic.

Naturally, even though we take the ideas developed by many mathematicians over many years for granted today, we can ask how these might have been discovered or invented originally. Although focusing on the breadth of the history of naive set theory would be a ridiculous task, spanning thousands of years, we can narrow down to the creation of more rigorous and developed set theory. And we find at the heart of this development to be one recurring and unique problem that fueled many discoveries and developments behind axiomatic set theory: the Continuum Hypothesis. At its simplest, the Continuum Hypothesis is the question of whether there could be a set with cardinality strictly between the set of integers and set of real numbers.

Georg Cantor is credited with the creation of this hypothesis, and so naturally he is mainly who we'll focus on in a discussion of it, but we find that Cantor's quest to prove this idea that he thought was so apparently true is one that takes his whole life, and multiple other mathematicians.

In the pursuit of making a linear narrative out of this journey, we begin with a discussion of Cantor's early work to explore how his ideas behind the Continuum Hypothesis were present for almost all of his career, and a surprisingly linear line of motivations for working on topics can be traced throughout his career. Of course, this discussion will be a simplified overview of Cantor's work, skipping over many results in favor of ones that contribute more readily to later ideas.

Then we have an overview of Cantor's establishment of set theory, again largely tailored around his statement and pursuit of the Continuum Hypothesis. Although he spent the rest of his career trying to prove the Continuum Hypothesis, he never did, and so from there we move on to developments of set theory after Cantor, which are again surprisingly motivated largely by the Continuum Hypothesis. A large element of this is the axiomatization of set theory, which is still used today, so we spend some time on this particularly.

Finally, we discuss the final results on the Continuum Hypothesis largely by Gödel and Cohen, proving finally that the Continuum Hypothesis is simply something that can be chosen to be true or false (within the current widely-accepted framework of mathematics). The actual proofs here are intense and require very specific high-level math, so this section will be the least-focused on the actual mathematics behind it, and more focused on the history behind these developments and the changes in overall mathematics brought on by them.

Throughout all of this, I hope to paint a broad view of how history influenced the math we take for granted today, and put these ideas that we might use interchangeably into a chronological order that gives surprising insight into each of their results. The mathematics that we have at any point is fundamentally human, and nowhere is this more apparent than in the history of how it was discovered throughout time: the discourse between people that creates conclusions built off of other mathematicians, or somehow creates new ones out of thin air by an especially creative thought. Without this human aspect of math, we would have none of what we do today.

2. Georg Cantor

Georg Cantor (1845–1918) was a German-Russian mathematician. He's known now for his set theory, which is an important topic in a discussion on the Continuum Hypothesis, but we should instead start with the beginning of his math career, since there end up being many parallels to later, and a fascinating progression that seemingly leads inevitably to the Continuum Hypothesis.

At the beginning of his research in math, he worked largely with number theory and analysis, making a couple of contributions to the fields. It wasn't that he would truly be set down the path of set theory until a colleague, Edward Heine (1821–1881) convinced him to join him on a functional analysis problem he'd had difficulty with.

2.1 Cantor's early analysis work

2.1.1 Uniqueness of Trigonometric Representations

The big question that he had been set upon by Heine was for any function that could be represented by a trigonometric series, whether that representation had to be unique. Heine had already proved uniqueness of trigonometric representation true for continuous functions of the form

$$f(x) = \frac{1}{2}a_0 + \sum(a_n \sin(nx) + b_n \cos(nx))$$

only when the series is uniformly convergent. But continuity and uniform convergence were restrictive conditions, so many others had been working at expanding these ideas into more general conditions.

In his proof of more general uniqueness, discovered in 1870, just a few months after being given the problem, Cantor first considered taking two trigonometric representations of the same function. If he subtracted these, giving a trigonometric representation equal to 0, he would just have to show that each term of this series would be forced to be 0, and thus the original "different" representations must actually be the same. He used this idea and other analysis work to show that this must be true, and published this uniqueness theorem.

The proof he discovered here generalized Heine's result since the only requirement was that the series was convergent for every value of x. But Cantor wasn't satisfied with this, and continued over the next couple of years working at generalizing it even further, first with series where there was a finite list of values where the series didn't need to converge (finite exceptional sets), and then naturally he began to consider the possibility of *infinite exceptional sets*.

To this end, Cantor worked at a more rigorous definition of the real numbers, starting with rationals, and creating irrationals from limits of convergent sequences of rationals, thus considering them as limit points of sets of rationals. He was with this able to build a definitive connection between these numerical values and the corresponding points on the real line, which led to his idea of the "point set," which would become the basis of his defining work. Although, at this point, it wasn't truly defined, and was instead employed in defining limit points, used to create an idea of derived sets that would also become integral to later developments.

2.1.2 Derived Sets as a Stepping Stone

Cantor defined a process of taking such a "set of points," P, and considering its set of limit points, P'. From here, one could continue this process any finite amount of times to find $P^{(v)}$ for some natural number v.

Cantor was able to use these to show that there were infinite sets P where $P^{(v)}$ had only a finite number of points, and therefore $P^{(v+1)}$ didn't exist. These were termed "derived sets of the first species," and these largely satisfied Cantor's desire for infinite exceptional sets for the uniqueness theorem, since he was able to show that uniqueness held when the set of exceptional points was any of these derived sets of first species.

But although Cantor's work on this uniqueness problem had been resolved to his liking, these new ideas, originally developed to prove this, seemed to become also important to him, as he developed them further.

In an important step toward his many ideas on the nature of infinity, Cantor decided to extend this process of deriving sets past a finite end. He considered sets P where $P^{(v)}$ existed for every finite v, so he defined ∞ to be *the first number past all finite numbers*, and defined sets where $P^{(\infty)}$ still existed to be "derived sets of the second species." This step of generalizing a process past finite ends into the realm of infinite ideas would fundamentally characterize many of his later important discoveries, including the Continuum Hypothesis, and in fact he would later consider his work on the infinite to be an extension of these derived sets.

2.2 Uncountability of the Set of Real Numbers

Richard Dedekind (1831–1916) was another German mathematician who was often in correspondence with Cantor. At one point after Cantor's development of derived sets, Dedekind criticized the idea, since if you take the set of rationals, the derived set would be real numbers, but any more deriving would just give the real numbers again.

Instead of listening to this criticism, Cantor instead continued down the path his work had sent him, and soon began to consider whether there might be a correspondence between the set of integers and the set of real numbers.

Soon after the letter to Dedekind asking about the relationship between integers and reals, he discovered that it was impossible to enumerate the real numbers with integers. Cantor's proof of the uncountability of the real numbers is a well-known result now, and often associated with his diagonalization argument. Interestingly, though, this was not the first proof Cantor developed to show this result, which instead largely used the ideas behind his derived sets:

Proof. Cantor assumed for contradiction that the set of real numbers was countable, in other words, that it was possible to list all real numbers as $\omega_1, \omega_2, \omega_3, \dots$ without leaving any out. Then he let α and β be any real numbers with $\alpha < \beta$, and could use these to show that there was at least one real number γ that wasn't included in the original list. Since this list claimed to enumerate every real number, this contradiction would prove that it wasn't possible for them to be enumerated in the first place.

To this end, he considered a process which was very similar to his derived set process. He would take the open interval (α, β) and find the first real numbers in the list just inside of this interval, α' and β'. Then he could do the same with (α', β'), continuing on until some $(\alpha^{(v)}, \beta^{(v)})$.

Then if this process was finite, there was only one element of the list left within $(\alpha^{(v)}, \beta^{(v)})$. So since there would still be more real numbers in the interval for any non-equal real numbers, if he took γ as any other real number in that interval, the proof was done.

So he considered the case of the process being infinite. The first subcase here would be if $\alpha^{(\infty)} < \beta^{(\infty)}$. Then very similarly to the last case, there must be another real number in $(\alpha^{(\infty)}, \beta^{(\infty)})$, so taking γ as any of these satifies the proof. In the last case, if $\alpha^{(\infty)} = \beta^{(\infty)}$, Cantor was able to show that $\gamma = \alpha^{(\infty)} = \beta^{(\infty)}$ was the necessary number not included in the list, since for sufficiently large v, $\gamma \notin (\alpha^{(v)}, \beta^{(v)})$, which made this case impossible in the first place.

Then in every case, it was impossible for $\omega_1, \omega_2, \ldots$ to list every real number, so the real numbers could not be countable. □

Cantor would work at streamlining this proof, perhaps his most popular, for many years, with another version using an idea of topological density, and another using the diagonalization argument. Cantor seems to have been fascinated by this result, and other ideas of infinity, since he decided to devote much of the rest of his career to ideas of the infinite. In order to do this, though, he realized he first needed to develop a more rigorous theory behind his "sets."

2.3 Cantorian Set Theory

From the first sentence of a later text on set theory called the *Beitrage*, Cantor defined his sets: "By a 'set' we mean any collection M into a whole of definite, distinct objects m (which are called the elements of M) of our perception or of our thought."

It was clear from this that Cantor realized he needed to essentially start from scratch, in order to develop the tools that would be necessary to more rigorously discover ideas about the infinite. Although today this effort would still be considered naive set theory, this was a large step forward, as he first tried to define a set, and then built out the rest of the results of the book from that, which would largely mirror the more rigorous axiomatization prompted later.

One of the new concepts developed by Cantor in the realm of the infinite built off of the idea of ordinal numbers and cardinal numbers. This is something we don't often have to think about, since finite natural numbers implicitly play both roles at the same time. Ordinal numbers are used to indicate the positioning or order of the numbers, or whatever they are representing. Cardinal numbers are used to indicate the size or cardinality of a set. Cantor would soon discover that cardinal numbers and ordinal numbers diverged when he began considering numbers representing infinities, which he called *transfinite numbers*.

2.3.1 Transfinite Ordinals

Cantor first considers the natural numbers, and notes how this set consists of repeated addition of units, which he calls the first principle of generation. The process by which Cantor realizes transfinite numbers is one that we've seen multiple times here: he generalizes this process further into the infinite. To get the first transfinite ordinal, ω, consider the first number after the entire set of natural numbers.

Then he reasons it would make sense to consider the first number after this, $\omega + 1$. In terms of ordinal numbers, this is perfectly reasonable, but if considering cardinality, this is exactly the reason we need the distinction in the transfinite world.

If we consider as Cantor does any ordinal number to be a set containing every previous ordinal number, then we can think of ω as $\{0, 1, 2, \dots\}$. Then $\omega + 1$ would be $\{0, 1, 2, \dots, 1\}$. This is reasonable and distinct in the realm of ordinals, but if we consider the cardinality of $\omega + 1$, we know we can rearrange this set so that we're instead considering $\{0, 1, 1, 2, \dots\}$, which must have the same cardinality as the natural numbers.

Continuing in the idea of being ordinals, we can find $\omega + 2, \omega + 3, \dots, \omega + \omega = \omega \cdot 2, \dots, \omega \cdot \omega$, and so on.

The idea of taking a succession of ordinals which technically has no end, and then simply "jumping ahead" to the first ordinal after all of those is what Cantor called the *second principle of generation*. Then you can use any combination of these two principles of generation starting with ω to reach any transfinite ordinal number.

2.3.2 The Continuum Hypothesis

Naturally, Cantor began to wonder about the size of the set of transfinite numbers compared to the set of natural numbers.

He first defined the collection of all finite whole numbers to be the *first number class (I)*. Then he defines the *second number class (II)* as the collection of all transfinite ordinals formed by the two principles of generation, starting with ω.

In a process similar to his uncountability proof of reals, he was able to prove that the cardinality of (I) was strictly less than that of (II), and furthermore that there was no cardinality between these sets, that (II) had the very next cardinality higher than (I). He saw the connections between this and his previous work, and decided that it was likely that (I) corresponded to natural numbers and (II) corresponded to real numbers, so **that there was no cardinality between the cardinalities of the natural numbers and real numbers**.

This hypothesis was the first, though not fully realized yet, statement of the Continuum Hypothesis. He thought that it would come naturally that there was a correspondence between these classes with $\mathbb{N}$ and $\mathbb{R}$, and made many claims about how he would soon prove this conclusively. Cantor throughout his life would be certain that there could be no set of size strictly between that of $\mathbb{N}$ and $\mathbb{R}$, and almost all of the rest of his career was dedicated to building more tools trying to prove this statement.

2.3.3 Transfinite Cardinals

The next important idea Cantor built, also attempting to work at the Continuum Hypothesis, was transfinite cardinals, in opposition to transfinite ordinals.

One of his first definitions of cardinality is that two sets have the same cardinality if there is a one-to-one correspondence between them.

So any finite set has a finite cardinality, and so he considers the first transfinite cardinal number to be the cardinality of the set of all finite cardinal numbers, called $\aleph_0$. As discussed above, many transfinite ordinals would have the same cardinality, and we find that

$$|\omega| = |\omega + 1| = \cdots = |\omega \cdot 2| = \cdots = \aleph_0.$$

So naturally the next question would be how to build higher cardinal numbers after $\aleph_0$. Cantor proved that if you took a cardinal to the power of 2, it would always be strictly larger than the cardinal, i.e. $2^{\aleph_n} > \aleph_n$.

Cardinals were useful in more simply stating his previous realizations on the topic, so he discovered that the cardinality of the class of countable transfinite ordinal numbers, (II), must be the very next cardinal number after $\aleph_0$, called $\aleph_1$. He was also able to show that the cardinality of the continuum is $2^{\aleph_0}$, which was another way of seeing uncountability of $\mathbb{R}$, since he had already shown that $2^{\aleph_0}$ must be strictly larger than $\aleph_0$.

Using these ideas of transfinite cardinal numbers, we're able to state the Continuum Hypothesis in a much more succinct way:

$$2^{\aleph_0} = \aleph_1.$$

This is the statement that Cantor spent his life trying to definitively prove, convinced for the rest of his career that the statement was true but a proof eluded him. He never was able to make a definitive statement on the Continuum Hypothesis. Georg Cantor died in 1918 never knowing what would eventually be discovered about his enduring hypothesis.

3. Popularization and Paradoxes

If Cantor was the only one interested in this problem, his death would have been the end of the story, and we never would have discovered the truths behind the Continuum Hypothesis. Luckily the problem was picked up by fellow German mathematician David Hilbert (1862–1943), who was a proponent of Cantor's set theory and transfinite numbers. Hilbert published a list of 23 unsolved problems in math in 1900 that would have powerful ramifications in mathematics if solved, and as the first problem Hilbert decided to put the Continuum Hypothesis. This list was very popular, and incited many mathematicians to work on these problems in an effort to develop math further, and as the first problem on the list, the Continuum Hypothesis got much more exposure than it had had before.

Hilbert was also an advocate for formal axiomatic systems, and wanted each of these problems to be founded in such a system, and wanted a concrete proof or counter-example for each statement. This was under the belief that every true statement within an axiomatic system must have a proof within that system, which we will soon see thanks to Gödel, is not true for every statement.

Meanwhile, other mathematicians were also considering Cantor's set theory overall, and finding ways to use the vagueness of its statements to show it wasn't always entirely self-consistent.

Cesare Burali-Forti (1861–1931) was one such mathematician, who thought of the famous Burali-Forti paradox. He considered Ω, the ordinal number greater than all the ordinal numbers. But then since Ω is itself an ordinal, it would have to be greater than itself, which is inherently a contradiction.

Bertrand Russell (1872–1970) was another mathematician who thought of a paradox within the loose definitions of Cantor's set theory. Russell's paradox simply constructs the set of all sets that are not members of themselves. Then this set isn't a member of itself, but then it must be in the set, so it must be a member of itself, but then it can't be in the set, and this continues forever.

Some mathematicians were adamant that these paradoxes completely dismantled all of Cantor's set theory, and that it was all useless now, and others argued that it only contradicted certain properties, or even that the paradoxes were making incorrect assumptions implicitly. There was plenty of back-and-forth on this, but to some mathematicians, it was clear that the best way forward was to formally axiomatize set theory so that any inconsistencies like these simply couldn't exist within such a framework in the first place.

4. Zermelo-Fraenkel Set Theory (with Axiom of Choice)

Ernst Zermelo (1871–1953) was one of these mathematicians who decided to axiomatize set theory, and his list of axioms was the one that caught on best. Zermelo listed 7 axioms which were able to define an entire base of set theory.

Abraham Fraenkel (1891–1965) was another mathematician who later noticed that Zermelo's axioms couldn't prove the existence of certain sets and cardinal numbers that were important to some mathematics, so he amended one of the axioms and added another.

Together, this set of axioms laid out by Zermelo and Fraenkel is known as Zermelo-Fraenkel Set Theory. It would later be shown that the Axiom of Choice was independent of these axioms, meaning it was consistent with them whether it was considered to be true or false, and so ZFC is what we call Zermelo-Fraenkel set theory with the Axiom of Choice included. ZFC is one of the most popular axiomatized set theories used today as the implicit or explicit basis of much of the mathematics worldwide.

We'll now look at the axioms behind ZFC. These are each written entirely in a string of a couple of different logical symbols, and together they implicitly define important concepts such as sets and membership. Technically there are multiple equivalent ways of writing out the axioms, so we use the one that Paul J. Cohen uses in the book in which he published his result on the Continuum Hypothesis (which will be the last thing we look at later on).

4.1 Statement of ZFC Axioms

1. *Axiom of Extensionality*

$$\forall x, y(\forall z(z \in x \Leftrightarrow z \in y) \Rightarrow x = y).$$

   This essentially says that two sets are equal if they have the same elements.

2. *Axiom of the Null Set*

$$\exists x \; \forall y(\sim y \in x).$$

   This axiom gives the existence of a set that has no elements in it, which is the empty set/null set, $\varnothing$.

3. *Axiom of Unordered Pairs*

$$\forall x, y \; \exists z \; \forall w(w \in z \Leftrightarrow w = x \vee w = y).$$

   This axiom allows the creation of another set from two sets by including them both in a set.

4. *Axiom of Union*

$$\forall x \; \exists y \; \forall z(z \in y \Leftrightarrow \exists t \; (z \in t \wedge t \in x)).$$

   This gives the existence of a set that is the union of all sets contained in another set. Together with the previous axiom, these allow for the union of any two sets to exist.

5. *Axiom of Infinity*

$$\exists x \; (\varnothing \in x \wedge \forall y \; (y \in x \Rightarrow y \cup \{y\} \in x).$$

   This axiom is the basis of the creation of whole numbers. It says any ordinal number will have a successor that is the union of it and the set containing it. It also allows for induction, as it says any set that contains the empty set and an ordinal will contain all successors.

6. *Axiom of Replacement*

$$\forall t_1, \ldots, t_k \ (\forall x \ \exists! y \ A_n(x, y; t_1, \ldots, t_k) \Rightarrow \forall u \ \exists v \ B(u, v))$$

$$\text{where } B(u, v) \equiv \forall r \ (r \in v \Leftrightarrow \exists s \ (s \in u \wedge A_n(s, r; t_1, \ldots, t_k))).$$

This axiom says that if there is a statement that uniquely describes a set as a function of a given set, then the image of any set under this function is itself a set. This axiom is carefully constructed to constrict statements that define sets, so that sets can't be defined by ridiculous statements like they were in the paradoxes.

7. *Axiom of the Power Set*

$$\forall x \ \exists y \ \forall z \ (z \in y \Leftrightarrow z \subseteq x).$$

This axiom allows the creation of the power set, the set of all the subsets of a given set.

8. *Axiom of Regularity*

$$\forall x \ \exists y \ (x = \varnothing \vee (y \in x \wedge \forall z \ (z \in x \Leftrightarrow \sim z \in y))).$$

This axiom states that every nonempty set has a minimal element with respect to membership. This ends up prohibiting sets from being members of themselves, which prevents paradoxes that include self-reference.

9. *Axiom of Choice*

$$\text{If } \alpha \to A_\alpha \neq \varnothing \text{ is a function defined for all } \alpha \in x, \text{ then there exists}$$
$$\text{another function } f(\alpha) \text{ for } \alpha \in x, \text{ and } f(\alpha) \in A_\alpha.$$

The Axiom of Choice allows that for any set of nonempty sets, we have a choice function that maps each member set to an element of that set. This axiom is important to many areas of math, and has multiple equivalent statements, such as "Given any family of nonempty sets, their Cartesian product is a nonempty set." Recall that this axiom is independent of Zermelo-Fraenkel set theory, so its inclusion, which is often done today, makes it ZFC instead.

4.1.1 ZFC Effects

These axioms alone are the entire assumed basis used to build up to much of modern math. ZFC is the standard set theory today, though there are alternatives, but most of math is built off of these founding axioms. So far, ZFC has worked very well, with no inconsistencies found, but we now know that we cannot prove the consistency of ZFC within the system itself. This result would be thanks to another mathematician who would make a similar contribution to the Continuum Hypothesis.

5. Independence of the Continuum Hypothesis

Kurt Gödel (1906–1978) was a German mathematician who greatly developed logic and math and the theory behind proofs that underlies all of math.

In his second incompleteness theorem published in 1931, he discovered that for any formal axiomatic system, which would naturally include ZFC, it is impossible to prove within the system that the system itself is consistent. Along with his first incompleteness theorem, this also showed that there were in fact statements that were true or false that could not be proven or disproven within a system.

Thus mathematicians discovered that axiomatizing and formalizing had prevented some paradoxes and inconsistencies, but they could never fully know every true statement in these systems.

Gödel made an even more important contribution to this topic in particular though, when he proved that the Continuum Hypothesis was consistent with ZFC. This was done by showing that it was impossible to prove within ZFC the negation of the Continuum Hypothesis. In other words, Gödel showed in 1940 that it wasn't possible to prove that there wasn't a set with cardinality strictly between $\aleph_0$ and $2^{\aleph_0}$ within ZFC. Thus it was possible that the Continuum Hypothesis was actually true as Cantor had hoped.

It took Paul J. Cohen (1934–2007) to finish this proof by also showing that it was also impossible to prove the Continuum Hypothesis within ZFC. This proof was highly technical, and developed a new set theory technique called "forcing," involving expanding a universe from an old one with a generic object, while specifically creating it in order to have certain properties. Using this technique, in 1963, Cohen was able to show that the opposite of the Continuum Hypothesis was also consistent with ZFC, or that it wasn't possible to prove there was a set with cardinality strictly between $\aleph_0$ and $2^{\aleph_0}$ within ZFC.

These results together provide the **independence of the Continuum Hypothesis from ZFC**, or in other words, that it is possible to assume the Continuum Hypothesis and still have a consistent system, but it is also possible to assume that the Continuum Hypothesis isn't true, and also still have another consistent system. Gödel also showed independence of the Axiom of Choice from ZF, which is why it had to be included in the axioms, and ZFC has to be clarified as distinct from ZF.

6. Conclusion

Cantor had specifically wanted the Continuum Hypothesis to be true, and Hilbert had specifically wanted it to be definitively proven or disproven. Although it's impossible to know what either thought about these results found decades after their deaths, I think it's safe to assume that Hilbert would be infuriated by the uncertainty Gödel discovered within mathematics. I think it's harder to tell what Cantor would have thought, though. Although it is possible to assume that the Continuum Hypothesis is true and use it in other results, Cantor spent much of his career convinced that it must always be true, and could be proven from other results, both things that we now know were not ever the case.

This result then might seem ultimately unsatisfying after all the work done to get to it. The definitive answer to the Continuum Hypothesis is that there is no definitive answer and you can do with it what you please. But I'd argue that this result is one of the most important in modern math, not necessarily because of what was ultimately discovered, but because of all of the math discovered and/or created in its pursuit. Consider all the many different mathematical concepts that had to be explained throughout this paper to begin to understand this topic, and also how most of it was developed specifically to try to discover this result. We learned that the real numbers are uncountable, about transfinite ordinals and cardinals, basic set theory, paradoxes, axiomatic set theory, the axiom of choice, incompleteness, forcing, and many other topics, all of which are extremely important to much of math today. Even if the conclusion of the Continuum Hypothesis's story wasn't entirely satisfying, the journey to get there should be satisfying enough on its own. And work is still developing, so it's more than possible that ZFC with the Continuum Hypothesis or with the opposite of the Continuum Hypothesis will fuel the next big discoveries in math.

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