A brief history of imaginary/complex numbers

Kyle Gomez

Abstract

Imaginary numbers are a difficult idea to comprehend, since they don't have as natural of an application to the real world as, for example, the counting numbers. Naturally, we ask, how did people discover complex numbers in the first place? And how did they develop over time into becoming another important and fascinating area of math, needed for many different applications? We look at ten people who shaped these discoveries, in one way or another, to bring complex numbers to where they are today.

1. Introduction

Starting off in math, most people learn the counting numbers first. Then how to do basic arithmetic with those numbers. Somewhere in here, we're presented with one of the first concepts we could possibly consider unintuitive: negative numbers. It's difficult to find a readily-apparent real-world justification for negative numbers, but we take them for granted nowadays, as something learned in elementary school math.

This instinctive rejection of the unintuitive marked much of the beginnings of math. Negative numbers weren't widely accepted in European mathematical spaces for many years.[1] There would be discovery and perhaps ocasional usage, but for such topics that mathematicians saw as useless or especially ones considered unintuitive, it was difficult to change public perception of these topics. It takes cooperation and working together to solve many mathematical issues, but if people can't generally agree that something is useful, it's even harder to make sure it seeps into the public consciousness.

Negative numbers were considered useless for a while, but some mathematicians persisted with them and continued working on the fringes, in the same way we see played out with many other topics originally considered non-practical. It was the work of these mathematicians that slowly brought negative numbers more and more into the forefront of math, and now they are commonly accepted globally.

Thinking back on the usual progression of learning math, usually after the basic arithmetic of addition, subtraction, multiplication, division, students begin considering exponentiation, as a progression of being repeated multiplication, itself repeated addition.

[1]Though in non-western spaces, these were enumerated possibly as early as 200 BC

There is a natural flow to these ideas, and then considering the inversion of exponentiation, usually starting with the square root. But suddenly there's an issue: students are warned not to use negative numbers in square roots.

In much the same way, early mathematicians encountered imaginary numbers, and - seemingly even more than negative numbers - were immediately turned away and disgusted with the thought. Early math was often dominated by an idea of geometrical interpretations; we see this with the Greeks especially, where their math isn't often done unless it's in geometrical pursuits. But geometry and complex numbers don't readily mesh well without many years of development and interpretation. So finding complex numbers was often off-putting to mathematicians concerned with geometry, often related to finding polynomial roots. If the graph doesn't cross 0 at the point, how could it possibly be a root??

When it came to complex numbers, disregarding, in much the same way as negative numbers, was the attitude for a long time. But just as described above, there were groups working in the fringes bringing these ideas to light more and more over time, until today where complex numbers are ingrained in our higher-level math, and applicable to many different areas of math.

In this paper, there are descriptions of ten different mathematicians throughtout the years who shaped this progression of how the world treated complex numbers, eventually informing us how we treat complex numbers today. Some of these people are the pioneers who blazed theory forward while contemporaries ignored their results. Some are the people who did the ignoring, but nonetheless had a hand in shaping opinions whether they wanted to at all. And one of them was even lost to time for many years. This is simply a small sample of the people who contributed to theories of complex numbers, but I believe the many interactions and developments made in this story are a microcosm for the development of all mathematics throughout time.

2. Heron of Alexandria

Heron of Alexandria (circa 60 AD) was a Greco-Roman mathematician and engineer known for creating many interesting inventions in antiquity, such as the first recorded steam-powered engine.

In the Greek tradition of working with geometrical shapes, one of his areas of research was formulas for frustums of pyramids, essentially pyramids with the top sliced off at some point, so that there are two square bases on the top and the bottom of the shape. See Figure 1.

Heron's brush with the imaginary came when he derived a formula for the height of a frustum based on the side lengths of the top and bottom squares, and the length of the

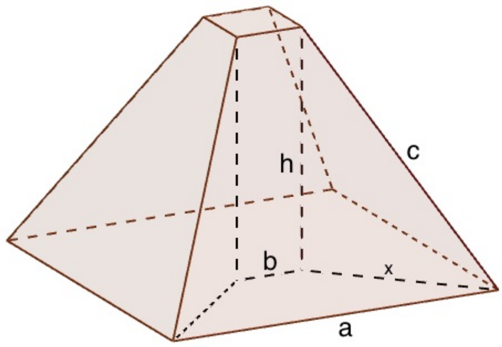

Figure 1: A frustum of a pyramid

diagonal from the top to the bottom:

$$h = \sqrt{c^2 - \frac{(a-b)^2}{2}}$$

Where a is the base length, b is the top square side length, and c is the length of the diagonal from corner to corner, yielding h, the height of the frustum.

In his *Stereometria*[2], the published formula for this includes the possibility for a square root of a negative number, in this case meaning such a frustum would be physically impossible. Heron even input such impossible numbers in his formula: using a base of 28, top side length of 4, and sloped edge of 15:

$$h = \sqrt{(15)^2 - \frac{(28-4)^2}{2}} = \sqrt{225 - 288} = \sqrt{-63}$$

So the formula gives the square root of a negative number! However, when Heron input these numbers, whether out of a mistake or confusion of the result[2] he recorded the result as $\sqrt{63}$ instead, which completely avoided the possibility of imaginary numbers altogether.[1] And so it wasn't for many more years that we have recorded evidence of someone actually finding complex numbers, and acknowledging the result.

3. Gerolamo Cardano

Gerolamo Cardano (1501-1576) was an Italian mathematician and scientist, who, among other accomplishments, introduced the binomial theorem to the West.

According to him, Nicolo Tartaglia, a close friend, shared with him that he had been the first to discover formulas for finding the roots of cubic and quartic equations. The only stipulation to seeing these formulas was for him to promise to never reveal them

[2] Some Greeks reportedly ignored or even abhorred the idea of irrational numbers (see rumors about Pythagoras), so it's possible imaginary numbers were entirely more inconceivable! Heron gives no acknowledgment, so we'll never know whether it was intentional or not.

to anyone. A few years later, Cardano discovered unpublished work of another mathematician, Scipione del Ferro, who had independently discovered very similar formulas. Thus Cardano felt justified in publishing the solutions[3] in his book *Ars Magna*[4], while also attributing them to del Ferro, Tartaglia, and a student of his, Ludovico Ferrari.[8]

Although it was possible for these cubic and quartic equations to yield complex solutions[4] Cardano never tried any such examples.

$$\begin{aligned} x^3 + px &= q \\ x^3 &= px + q \\ x &= \sqrt[3]{\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} \end{aligned}$$

Instead, in another part of the text, he proposed an exercise to find two numbers that add to 10 and multiply to 40.

$$x + y = 10, \quad xy = 40$$

He works this out as he would for any other pair of numbers, but discovers that the solutions here are $5 \pm \sqrt{-15}$. This becomes apparent during the derivation, and Cardano makes many references to the absurdity of this. He calls the exercise "impossible" and "mental torture" but nevertheless continues to find the solutions. Ultimately, after writing down the first known recorded reference to complex numbers, Cardano simply calls the result "as subtle as it is useless" and moves on to different topics.

Thus we see that even in the 16th century, many mathematicians found the idea of complex numbers to be completely useless, even while working with polynomial roots, a subject we now know is inevitably intertwined with complex numbers. Yet, while this was a common thought at the time, not everyone would dismiss complex numbers as Cardano did.

4. Rafael Bombelli

Rafael Bombelli (1526-1572) was an Italian mathematician known for simplifying algebra so that even people who weren't mathematicians could read through his book, *L'Algebra*[3].

In the course of the book - using names and words for numbers and concepts rather than the mathematical notation we know today - Bombelli takes the step forward that Cardano didn't.

He considers roots of polynomials, and finds complex roots, and decides not to dis-

[3] Whether he truly was justified is something he and Tartaglia would argue for years.

[4] In fact, eventually it would be shown that for many solutions, complex roots were necessary to the Fundamental Theorem of Arithmetic

regard them, but instead to study them further. Thus Bombelli becomes one of the first people in recorded history to begin developing properties and arithmetic for complex numbers. He noted that they were different from real numbers, and that when adding them, you have to treat the real part and imaginary part as separate. Bombelli also considered the different ways to multiply i and $-i$, using notation of calling what we know as i "plus of minus" and $-i$ as "minus of minus". He enumerates these properties of multiplication with imaginary numbers that we know today:

Bombelli's words *(translated from Italian)*	**Modern interpretation**
"Plus by plus of minus, makes plus of minus"	$1 \cdot i = i$
"Minus by plus of minus, makes minus of minus"	$-1 \cdot i = -i$
"Plus by minus of minus, makes minus of minus"	$1 \cdot -i = -i$
"Minus by minus of minus, makes plus of minus"	$-1 \cdot -i = i$
"Plus of minus by plus of minus, makes minus"	$i \cdot i = -1$
"Plus of minus by minus of plus, makes plus"	$i \cdot -i = 1$
"Minus of minus by plus of minus, makes plus"	$-i \cdot i = 1$
"Minus of minus by minus of minus, makes minus"	$-i \cdot -i = -1$

5. René Descartes

René Descartes (1596-1650) was a French philosopher and mathematicians. In the world of math, he's known for developing analytic geometry, combining the subjects of algebra and geometry, which were previously thought to be fairly separate ideas. He is perhaps more well known to history as a philosopher, with many published works on different topics, and he was the one to coin the phrase "I think, therefore I am."

Descartes, like many before him, continued to think in terms of geometry. Although he had closely connected algebra and geometry, he still couldn't find a geometrical approach to thinking of complex numbers. In this story, Descartes did not further the theory of complex numbers, but instead joined detractors in calling it useless. However, he did accidentally make an impressive contribution in his own way.

In *La Géométrie* [5], he was the possibly the first person to call the square roots of negative numbers "imaginary." This term promptly stuck, and we still call these numbers "imaginary" to this day. In the vein of being a detractor, Descartes had intended it as an insult to the complex numbers highlighting how worthless he thought they were, but it left a lasting impact on the topic nonetheless.

6. Abraham De Moivre

Abraham de Moivre (1667-1754) was a French mathematician, known for his work in probability theory, including developing the normal distribution.

In his article "De sectione anguli"[9] on arc lengths, he developed a formula from trigonometric identities which permit (with some substituting of certain trigonometric terms) what's known as the De Moivre Formula, involving a complex number

$$(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta)$$

His contribution to complex numbers is perhaps less than others on this list, but still important because it would set the stage for an implicit link between trigonometric identities and complex numbers. This would allow the next mathematician to take his formula to create one of the most famous formulas in all of math history.

7. Leohnard Euler

Leohnard Euler (1707-1783) was a Swiss mathematician, known for many things including founding graph theory and topology, and introducing much of the modern notation we still use today, among many other things.

He might be most well-known for the number e named after him, but he was also the one who popularized using i to denote the imaginary part of a complex number.

It was in his book *Intoductio in analysin infinitorum*[6] that he further built many important concepts about complex numbers. Using his number e and the natural log, he first defined exponentiation of complex numbers by:

$$\left(1 + \frac{z\sqrt{-1}}{i}\right)^i = e^z$$

It's important to note that he hadn't yet started using i for imaginary numbers, so the i here is meant to be "an infinitely large number."

Then in a later chapter, he works through some trigonometric properties concerning the unit circle. He first factors out the well-known property:

$$\begin{aligned} \sin^2(z) + \cos^2(z) &= 1 \\ (\cos(z) + \sqrt{-1}\sin(z))(\cos(z) - \sqrt{-1}\sin(z)) &= 1 \end{aligned}$$

This might be beginning to look familiar already, but Euler first has to do work with trigonometric series and infinite sequences to discover:

$$\begin{aligned} \cos(v) &= \frac{\left(1 + \frac{v\sqrt{-1}}{i}\right)^i + \left(1 - \frac{v\sqrt{-1}}{i}\right)^i}{2} \\ \sin(v) &= \frac{\left(1 + \frac{v\sqrt{-1}}{i}\right)^i - \left(1 - \frac{v\sqrt{-1}}{i}\right)^i}{2\sqrt{-1}} \end{aligned}$$

Since these include the exponential of complex numbers from before, he was able to substitute with that equation to get equations with e and sin and cos, and soon unveils the formula that tied complex numbers to trigonometry even more resolutely and simply, and ended up defining complex exponentiation altogether itself:

$$e^{v\sqrt{-1}} = \cos(v) + \sqrt{-1}\sin(v)$$

Euler's formula, still well-known to this day, is now a defining property of both complex numbers and trigonometric functions. It is likely his most important contribution to complex numbers, but since his contributions to math are so numerous it's hard to argue for this being his defining achievement overall. Nonetheless, this formula was still hugely important and continues to be to this day.

8. Caspar Wessel

Caspar Wessel (1745-1818) was a Danish-Norwegian cartographer and also mathematician. He was more predominantly a cartographer, and his mathematical observations were often made in pursuit of cartography.

While surveying once, he naturally wanted to make a coordinate system to show a plane that could correspond to the surface he was working on. He made the important leap that many had been searching for for years when he decided to use a real axis and an imaginary axis to build a complex plane to get this two-dimensional coordinate system. Before this point, it had been theorized that complex numbers could be used this way, but Wessel's journal is the first documented time someone ever truly discovered the complex plane as we know it now. This was finally the solid geometric interpretation that many early mathematicians thought complex numbers were lacking, and this is now largely the way complex numbers are fundamentally considered.

He soon published this idea: *Om directionens analytiske betegning*[11], which outlined the procedure he used to create the complex plane. The most interesting thing though, was that this paper was only published in Dutch, while the predominant mathematicians of the time were more often in France, Italy, Germany. These were other European countries with different languages, and although Latin was often used in mathematical writing to bridge the language gap,[5] Wessel's paper was only ever published in Dutch. So not many mathematicians would see it and its revolutionary ideas, and it never caught on like the rest of the ideas in this story.

It wasn't until almost a full century later, when a Danish graduate student in the 1890's wrote a dissertation about the history of Dutch mathematicians including Wessel. At this point, the complex plane had been discovered widely by others,[6] and complex theory had developed more significantly. The student hadn't noticed the profundity of his discovery of Wessel and his paper, but one of the mathematicians who read this dissertation realized

[5] Since of course, any respectable European mathematician would have to know Latin as a prerequisite.
[6] See the next section about Gauss for more on this.

quickly that this was an important piece of math history that had been lost to time. He quickly published the discovery on the front page of popular Dutch journals, which set off a chain of people discovering the result, and finally translating the original paper to make it widely available, and to bring Wessel's contribution to the forefront.[10]

9. Johann Carl Friedrich Gauss

Johann Carl Friedrich Gauss (1777-1855) was a German mathematician who proved the Fundamental Theorem of Algebra using complex numbers as an important aspect of the theorem, among many other mathematical achievements.

He developed the principles of complex numbers greatly. Some retrospectively consider his work to be the foundation for complex analysis that would soon be fully fleshed out by other mathematicians. Among his contributions was popularizing the phrase "complex numbers" which he says was because he found the "imaginary" terminology to be too off-putting. He argued that complex numbers were ultimately understandable and important, but calling them "imaginary" meant that many people still had a perceived idea of heightened difficulty and/or uselessness.[7]

In his paper "Theoria residurorum biquadraticorum. Commentatio secunda"[7], he published the idea of the complex plane. This was independent of Wessel[8] and any others who had similarly discovered this notion of a complex plane. The reason I bring it up here rather than mention the others before that weren't forgotten, is that Gauss's publication of this was the main factor in the popularization of the plane. For every important result, discovery is only one side of the story, but popularization is an important other side that sometimes comes well after discovery. We can give Gauss some credit for discovering it independently, but we must give him credit for getting the word out to the larger mathematical world.

10. Augustin-Louis Cauchy & Bernhard Riemann

Augustin-Louis Cauchy (1789-1857) was a French mathematician who made many important discoveries, including rigorously proving the basic theorems behind calculus.

Bernhard Riemann (1826-1866) was a German mathematician who also developed many important concepts including the Riemann integral and the Riemann hypothesis.

Grouping these two together is not done to diminish either of their work independently, but instead to gesture widely to the large group of contemporaries also working with these two. They might be some of the biggest names among this group, but there were many other important contributors, too many for me to list here.

[7] Descartes would probably be very happy about his effect on the public perception of "imaginary numbers" but I like to think he would be swayed to the other side by the many applications since discovered.

[8] Recall that at this time, Wessel's work was still largely lost to the wider mathematical world.

Together, Cauchy, Riemann, and a large group of contemporaries worked to develop the bulk of what we know today as complex analysis. They took everything from their predecessors that had been learned about complex numbers, and they really dove into what made them so different from real numbers, and how to fully develop our knowledge of their inner workings.

Admittedly, I don't know much about complex analysis, but even if I did, I believe that even an overview of what these mathematicians developed would not fit on this paper. Complex analysis has developed so far that it's now often a full course on its own in many colleges. This course brings together all the many years of contributions developed by the people in this paper, and so many more that didn't fit here, and goes even further with all of it to new heights.

11. Conclusion

As with many topics, we find that complex numbers were originally ignored or openly mocked and reviled by people who didn't notice how useful they could be. It was because of the work of (most of) these mathematicians that they could be brought into the limelight and really shine in their applicability to so many different parts of math. Like I've said multiple times, there are so many other people who made important contributions to this topic, and those who continue to make contributions to complex analysis today.

Piecing together this story and weaving it into a narrative was a great exercise in my understanding of math. Knowing the historical context for important discoveries really helps to make me feel like I can have a deeper appreciation for these insights and contributions made so long ago. It also adds such a human side to these formulas and equations that can seem so unforgiving and cold sometimes. It's profound to find that just this narrative thread I've decided to tell includes so many personal stories, from a dying oath possibly broken, to someone being rediscovered and reappreciated a century after their accomplishment.

It also really showed me the importance of publishing and making facts widely known. I found that discovery of an important topic and popularization of it were two completely different endeavors. I believe that we're lucky to have the internet now, so that such things can be spread much more quickly and widely, but I think there's a drawback to this too: there's just too much to sort through now. This was probably already the case for many back when it was just many journals and magazines publishing important papers, but it starts to feel like there's just too much information to have time to find the most important facts. Not to mention, the internet can start to mirror the old ways, by losing important facts to small corners soon inaccessible. Who knows what's out there that's been discovered mathematically but still can't find its way out of obscurity?

I appreciated the feedback I got for my presentation. It seemed that people enjoyed it and generally had the same inciting idea I had about how mathematicians first found these ideas. There was talk about how the same names kept popping up in many different subjects, and I really believe this has to do with the breadth these individuals covered in math. Whether they truly were "gifted" beyond ordinary circumstances is impossible to know, but many of them dipped into so many different branches and left their mark that no matter where in math we go, there's always a good chance of stumbling onto the same names. This could also correspond to the Pareto Principle, in that the top 20% of mathematicians discovered the top 80% of important concepts. I don't know the numbers on these for sure, but I wouldn't be surprised if it corresponded with it.

There was also talk about how euro-centric this narrative ended up being, and I really think that's just inherent to much of the history of math. Since Europe, especially in the Renaissance and Enlightenment periods, was instrumental in the formalization of math, we find that even though important concepts were being discovered in non-Western spaces, it took the formalism of Europe to make these concepts more widely accepted to the bulk of mathematicians, who were themselves Europeans. This absolutely also goes back to the point about publishing and making things well known is much different from discovery, and we have to mention of course that Europe was the region that sooner adopted and made widely available the printing press. This meant that for a large portion of history, Europeans were ahead in terms of how many of the same copy of a book they could print, and thus how much the idea in that book was able to spread. This combination of printing ability and emphasis on formalism and actually having a document is what I think makes much of math history so European-focused, but of course there could be many more factors at play that I haven't considered. I think this would be another interesting topic to look into further at some point.

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