Differential Equations

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October 2020

1 Introduction

Mathematically, differential Equations are equations containing derivatives. This paper will mainly focus on some applications of differential equations, mostly population models.

2 Some Related History

This concept of differential equations was come up with by Issac Newton, in section 2 of *Method of Fluxions*, 1671. He gave three different examples of differential equations.

$$\frac{dy}{dx} = f(x) \tag{1}$$

$$\frac{dy}{dx} = f(x, y) \tag{2}$$

$$x_1 \frac{\partial y}{\partial x_1} + x_2 \frac{\partial y}{\partial x_2} = y \tag{3}$$

The first two equations are ordinary differential equations and the last one is a partial differential equation. He came up with these equations when he was trying to solve the following problem: from the velocities of the motion at all times given, to find the quantities of the spaces described. This is the first application of differential equations, calculating physics problems, especially velocity and distance problems.

One other application I mainly presented is population models. In the late 18th century biologists began to develop techniques in population modeling. Differential equations are largely applied in this process. Among them, one of the most basic and milestone models of population growth is the logistic model of population growth formulated by Pierre François Verhulst in 1838. In my presentation, I started from there and introduced several population models.

3 Logistic Population Model

The logistic model of population growth was in a series of three papers by Pierre François Verhulst in three papers from 1838 to 1847.

Here is an example where this model can be applied. Suppose there are rabbits living on a grass field. Their number started out small but grew really quickly. As each rabbit would give birth to babies, reproduction is generally considered to be consistent with the current population, P. However, there is only a limited amount of grass and it can only supply N rabbits. So it would stop growing when P gets closer to N.

There are two assumptions when we devised this model.

- The reproduction rate is proportional to the size of the population when the population size is small
- The growth is negative when the size gets larger than a certain number.

Based on these assumptions, we came up with a model.

$$\frac{dP}{dt} = kP(1 - \frac{P}{N}) \tag{4}$$

Here P stands for population. N stands for the upper bound. k is the growth rate, and t is just time. When P is small, $(1 - \frac{P}{N})$ is basically 1 and can be ignored. The growth rate is kP, proportional to P. However, the closer it gets to N, the slower it grows. It would be extremely close to N but not reaching there.

When P equals to 0 or N, it reaches an equilibrium as the growth rate is 0. It just stays the same.

With this model, the initial stage of growth is approximately exponential (geometric); then, as saturation begins, the growth slows to linear (arithmetic), and at maturity, growth stops. This can be shown in the following Figure 1. Here I set k to be 1 and N to be 50. The growth trend is very well shown in the figure.

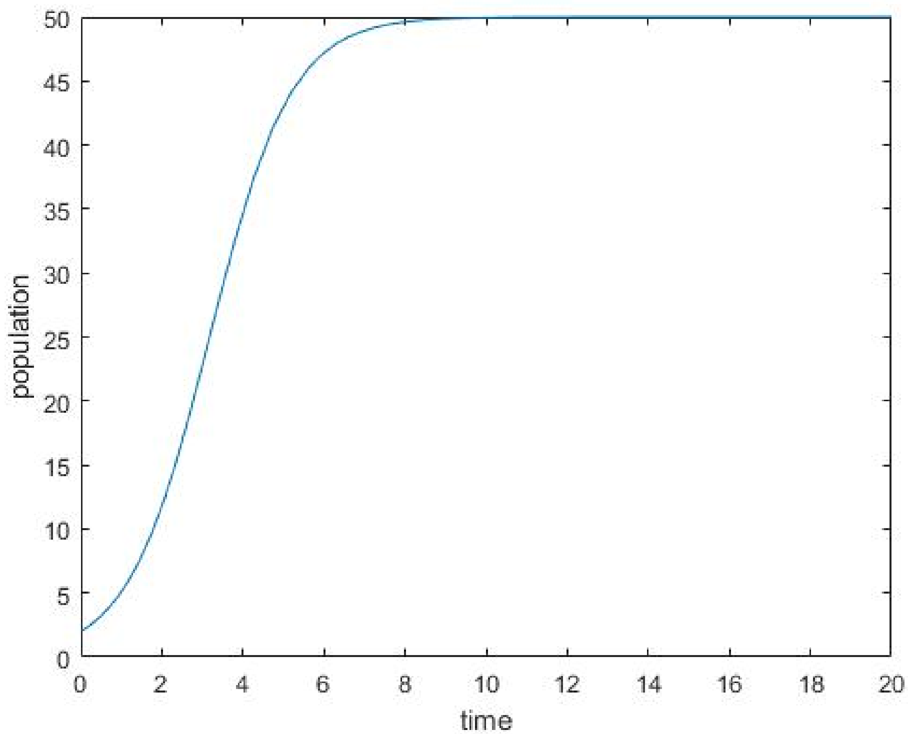

Figure 1: Logistic Population Model

4 Predator-Prey Model

Here we are faced with a new example. Instead of modeling the population of a single variable, this time, we are dealing with two variables. This would be an ODE group, formed with multiple ordinary differential equations.

This model models the population change of fox and rabbits. Here, fox, F, eats rabbits, R. This example takes place in the following conditions.

- With more foxes and more rabbits, more rabbits die.
- With more foxes and more rabbits, more foxes are born.
- Rabbit birth rate not related to fox in this model.
- Fox death rate not related to rabbit in this model.

The model goes as following.

$$\frac{dR}{dt} = aR - cRF, \tag{5}$$
$$\frac{dF}{dt} = -bF + dRF \tag{6}$$

The variables, F stands for fox and R stands for rabbits. t is time. The parameters have the following meanings.

- a – birth rate of rabbits
- b – death rate of fox
- c – death rate of rabbits, interrelated with both rabbit and fox
- d – birth rate of fox

The great thing about differential equation models is that it is straightforward and we get to see a lot of interrelations even without actually solving the equation.

First, let's check the equilibrium solution. The equilibrium equations go like this.

$$0 = aR - cRF, \tag{7}$$
$$0 = -bF + dRF \tag{8}$$

There are two cases.

First case: $F = 0$, $R = 0$

Second case: $F = \frac{a}{c}$, $R = \frac{b}{d}$

From this result we can deduce that in nature, a stable prey-predator system, probably results F and R stays in the second case. Since in the second case, they would remain unchanged.

If one of them extincts, the population change of the other specie can also be deduced. If F goes to 0, $R = R_0e^{at}$, the rabbits would just grow exponentially if foxes are gone. If R goes to 0, $F = F_0e^{-bt}$, resulting in a quick decline in fox population.

There are many other analysis methods that can applied to this. So this is the predator-prey model. It involved two species that are interrelated. Now we are going to introduce some real-life model that are much more complicated, but based on similar ideas with the former two models.

5 Blue Crab Population Model

Here in this section some models of blue crabs is going to be introduced. They are from the experiment I did in the summer.

5.1 Introduction

The official name of blue crabs is *callinectes sapidus.*They are a type of crabs, living along the western edge of the Atlantic Ocean and the Gulf of Mexico.

5.2 Background of Fangming's model

First we introduce Fangming's original model of the blue crab population.

Before introducing the model, a few facts about blue crabs and the model need to be stated.

First, there are three phases in the lifecycle of blue crabs, larvae, juvenile crabs, adult crabs. Larvae will not be modeled. Fangming modeled a juvenile class, J, and an adult class. A. Along with time, t, these are the three variables in the equation.

Second, cannibalism exists in blue crabs and adult crabs would hunt juvenile crabs.

Third, adult crabs are exposed to fishing while juvenile crabs are not fished. However, juvenile crabs that reach a certain age would be able to avoid cannibalism and become targets of fishing.

5.3 Fangming's Model

$$\frac{dJ}{dt} = \frac{\alpha A}{1 + bA^2} + (p + A)\frac{k_{max} J^2}{x^2 + J^2} - \beta J, \tag{9}$$

$$\frac{dA}{dt} = \beta J - mA - F(A) \tag{10}$$

This is the model, from *Stage-structured Blue Crab Population Model with Fishing, Predation and Cannibalism*, by Fangming Xu, 2020. It is actually formed of these parts.

Juvenile Crabs = Recruitment(from reproduction) - (Cannibalism + Predation) - Maturation

Adult Crabs = Maturation - other mortality - Fishing

5.4 Explanation and thoughts

Here we get to see the convenience of differential equations. Different factors affecting the population can be linearly included. This model has the potential of containing lots of information and factors, all influencing the population.

Let's take a look at how each condition is shown in the equation.

First, the three phases, or two phases since Larvae is not modeled, is very well shown in the equations as juveniles, J, and adults, A, are separately modeled. As time goes, some J would lose due to maturation and A would increas because of that. Natuarally, these two terms share the same parameter β, only one being negative, one being positive.

Second, cannibalism is shown in the second term of Equation (9). We can see that A is included, showing the adults are involved in the predation process.

Third, in the third term of Equation(10), adults are fished. This term do not exist in the previous equation because juveniles are not fished. It can be very well represented.

5.5 graphs of Fangming's model

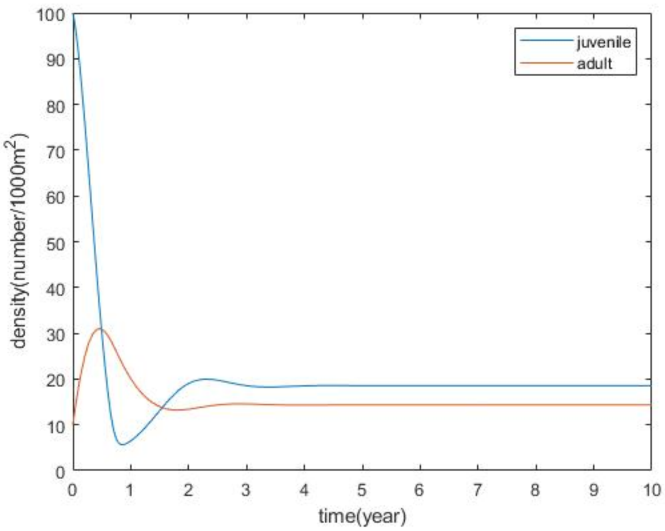

Figure 2: Fangming's Model

Here, blue line stands for juveniles and red line stands for adults. We can see the process of them coming to an equilibrium.

5.6 Table of parameters for Fangming's Model

Parameters	Meaning	Unit	Value
α	Maximum per capita reproduction rate	y^{-1}	30.62
b	Decrease in reproduction due to competition	number$^{-2}10^6 m^4$	0.01256
p	Density of predators	number $10^{-3}m^{-2}$	10
k_{max}	Maximum feeding rate	y^{-1}	Equation(3)
x	Density of prey at half of the maximum feeding rate	number $10^{-3}m^{-2}$	
β	Maturation rate of juveniles	y^{-1}	1.09
m	Adult mortality rate	y^{-1}	0.9
s	Density of adults at half of the maximum fishing rate	number $10^{-3}m^{-2}$	69.130

Table 1: Fangming's Parameters

5.7 Hematodinium Model

Hematodinium(*hematodinium perezi*) an internal dinoflagellate parasite that infects crustaceans, including blue crabs. We were working on including this disease factor in our current model in the summer. However, the research stopped and it is currently unfinished.

I mentioned and described it in the presentation. But since it is unfinished,it would be better not to include it in the paper.

So, these are the contents of my presentation.

6 Peer Feedback

As Professor Li has said, modeling is more about guessing in the first stage. Based on some individual experimental results, or similar models of other species, we need to guess the form of the model that can describe the relationship the best. Sometimes it takes the simplest model to describe a seemingly complicated relationship. This is a fascinating process, full of uncertainty.

Also, I have received some feedback regarding my microphone being bad. It seems that in many cases, a large factor that affects your presentation, is the devices supporting it. I will put that in mind and do something about my devices next time.

7 Conclusion

Overall, differential equations have so many applications and I introduced some population models that rely on it. There are many advantages of this equation form. It is no surprising that it is used in all sorts of fields.

8 Reference

Method of Fluxions by Issac Newton, 1671

https://en.wikipedia.org/wiki/Population$_{m}odelHistory$

https://en.wikipedia.org/wiki/Logistic$_{f}unctionHistory$

Stage-structured Blue Crab Population Model with Fishing, Predation and Cannibalism by Fangming Xu, 2020