Turing Patterns

Abstract: Turing patterns, first proposed by mathematician and computer scientist Alan Turing in 1952, describe a fascinating phenomenon of pattern formation within biological systems and beyond. His model answers questions like "How do zebras get their stripes?" or "How do cheetahs get their spots?" This paper provides a comprehensive overview of Turing patterns, beginning with Turing's idea and the theoretical framework of his reaction-diffusion equations. Then, we will discuss how patterns form from a homogeneous system and the mechanisms behind which patterns will develop. Next, we introduce Turing's original mathematical model and how each parameter affects the formation of patterns, or lack thereof. Furthermore, we explore models based on Turing's original model that were created after his death. Additionally, we highlight recent advances in understanding and modeling Turing patterns in both biological and non-biological systems. Lastly, we reflect on the presentation and further research to be completed.

1. Alan Turing and Morphogenesis

1.1: **Alan Turing**. Alan Turing was a well-known English mathematician and computer scientist in the mid-1900s. He is best known for his role in the development of theoretical computer science with the Turing machine. Additionally, during World War II, he worked with a team to create the Bombe, an electrochemical device designed to decipher encrypted codes sent by the German military using the Enigma machine. Turing's expansive work laid the foundation for modern day computing and artificial intelligence. In 1952, Turing was convicted of committing homosexual acts and was forced to undergo hormone treatment, otherwise known as chemical castration. Shortly thereafter, in 1954, Turing died at the age of 41 after eating an apple poisoned with cyanide. There are conflicting reports on whether he committed suicide or was poisoned accidentally. [1]

1.2: **Turing and Mathematical Biology**. One of Turing's more unrecognized interests during his studies was his contribution to theoretical biology. In 1951, he published a paper titled *The Chemical Basis of Morphogenesis*. In this paper, he explored how patterns in nature, such as the spots and stripes that can be found on animal skin, naturally arise. Morphogenesis is defined as "the development of patterns and shapes in biological organisms" [1]. Turing argued that these patterns are a result of morphogens: signaling molecules whose non-uniform distribution governs the pattern of tissue development in the process of morphogenesis [8]. These morphogens diffuse, or spread out, from a localized source to create a concentration gradient. This chemical gradient is the basis for the mechanism behind Turing patterns.

1.3: **Reaction-Diffusion Systems**. A reaction-diffusion system is the interaction between local chemical reactions in which substances are transformed into each other and diffusion causing the substances to spread out over a surface in space. This process is the basis of Turing's pattern formation theory. During the time of Turing's discoveries, many believed that diffusion created

stable conditions because diffusion is a dissipative system. As a result, it was thought that diffusion would simply cause substances to break apart and scatter. Instead, Turing proposed that diffusion actually destabilizes a chemical system, causing the creation of patterns. In Turing's model, a system begins with two chemicals: an activator and an inhibitor. The activator promotes the production of both itself and the inhibitor. On the other hand, the inhibitor halts the production of the activator, creating a negative feedback loop. As the inhibitor stops the activator, it consequently stops the production of more of itself. The inhibitor plays a critical role in controlling the spatial spread and stability of emerging patterns. [9]

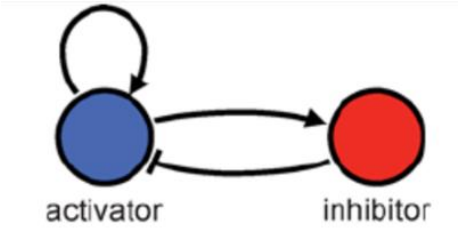

Figure 1: Activator-inhibitor loop

1.4: **Reaction-Diffusion System Analogy**. Imagine there is a cheetah with no spots (Figure 2). Now picture its brown fur as a dry forest. Throughout this forest, fires break out. Luckily, though, there are firefighters stationed all throughout the forest who work quickly to extinguish the fires by surrounding them and putting them out with water. Additionally, the firefighters can always move faster than the fire, allowing them to keep the fires under control. Once the firefighters put out a fire, there is a charred spot remaining in the forest and they move on to the next one (Figure 3). Zooming back out to the cheetah, it now has small black spots all over its previously plain fur (Figure 4).

Figure 2: Analogy - cheetah with no spots

Figure 3: Analogy - firefighters extinguishing fires throughout the forest

Figure 4: Analogy - cheetah with spotted pattern

The fire can be thought of as the activator and the firefighters as the inhibitor, both diffusing throughout the forest. The fire creates more of itself as well as draws in more firefighters. The firefighters work to put out the fires, but as more and more fires are put out, less and less firefighters are needed. It is important to note that the inhibitor must always move faster than the activator, or else the activator would spread out of control and overtake the whole system. By adjusting the rates at which these two components spread, different patterns can form by Turing's rules. [10]

2. Formation of Patterns

2.1: **Turing Instability**. Every system originally begins in a homogenous state where the concentrations of activator and inhibitor are uniform. Turing instability is the phenomenon in reaction-diffusion systems where homogenous states become unstable and spontaneously give rise to spatial patterns. To begin the process, there must first be some kind of small disturbance, or perturbation, to the system that destabilizes the homogenous state. As a result, the activator and inhibitor concentrations become non-uniform and begin to act on each other as described above. The instability escalates from the interplay between the reaction and diffusion processes in the system. As the perturbations grow, they eventually reach a size where they become self-sustaining and promote the formation of spatial patterns. Pictured below are examples of natural patterns found on animals that are formed through this process. [3]

Figure 5: Spotted cheetah print

Figure 6: Striped pattern on zebra

Figure 7: Labyrinth pattern on Mbu pufferfish

2.2: **Stripes vs. Spots**: Even though stripes and spots are seemingly different, they are formed by the same mechanism. One way that stripes form instead of spots depends on the rate by which both the activator and inhibitor diffuse throughout the system. As stated previously, the inhibitor must always diffuse faster than the activator. However, if the activator diffuses at a comparatively quick pace while still being slightly slower than the inhibitor, it can temporarily "outrun" the inhibitor. As the activator continues to diffuse, the inhibitor "chases" it, blocking it from all but one direction. Eventually, though, the inhibitor will catch up and ultimately cut off the activator, leaving a stripe in its wake. One can also consider a pattern of spots that begin to spread out and "leak" into each other, creating a stripey or labyrinth pattern (Figure 6). Alternatively, when Turing patterns play out on irregular surfaces like an animal's body, different patterns can arise on different parts. The exact same system can create spots on a larger surface and stripes on a small one. For example, a cheetah often has spots all over its body and stripes on its tail since its tail is a much smaller surface. [10]

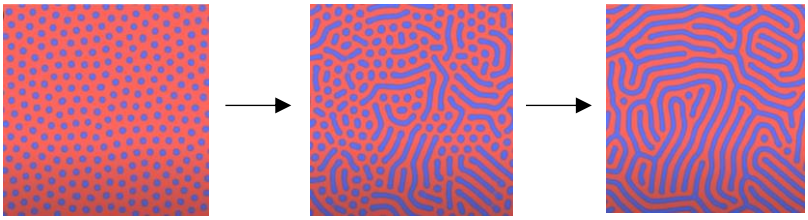

Figure 6: Spots "leaking out" into a labyrinth pattern

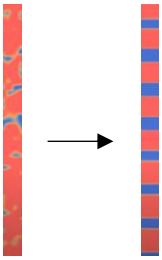

Figure 9: The formation of stripes on a small surface

3. Turing's Equations

3.1: **Partial Differential Equations**. A partial differential equation, or PDE, is defined as "a mathematical equation that involves two or more independent variables, an unknown function (dependent on those variables), and partial derivatives of the unknown function with respect to the independent variables" [4]. In other words, a PDE computes a function between various partial derivatives of a function with multiple independent variables – where a partial derivative of a function with several variables is its derivative with respect to only one of the variables while the others are held constant. Partial derivatives are denoted with the symbol "∂". For example, a partial derivative written as $\partial f/\partial x$ can be read as: the partial derivative of function f with respect to x while y is held constant (assuming that the function f only has independent variables x and y).

3.2: **Turing's PDEs**.

$$\frac{\partial u}{\partial t} = D_u \nabla^2 u + f(u, v)$$

$$\frac{\partial v}{\partial t} = D_v \nabla^2 v + g(u, v)$$

The partial differential equations that Turing created to explain Turing patterns are shown above where:

- u and v are the concentrations of the activator and inhibitor substances, respectively.
- D_u and D_v are the diffusion coefficients.
- ∇^2 is the Laplacian operation, representing the spatial variation or curvature.
- $f(u,v)$ and $g(u,v)$ represent the reaction terms, describing how the concentrations of u and v change due to chemical reactions.

Changing these parameters affects the pattern formation. [3]

3.3: **Changing Parameters**. The diffusion coefficients D_u and D_v describe how fast the activator and inhibitor move throughout the surface. Increasing these coefficients can lead to the formation of more homogenous patterns whereas decreasing them can promote the formation of more localized patterns with sharper transitions between areas of higher and lower concentrations. The reaction rates *f(u,v)* and *g(u,v)* describe the production of both the activator and inhibitor and how the two interact with each other. Increasing the rates of activator production or inhibitor inhibition may lead to the formation of more pronounced patterns whereas decreasing these rates can result in the suppression of pattern formation or the creation of simpler patterns. The initial conditions describing factors such as the surface and the beginning concentrations of activator and inhibitor alter the starting point from which the system evolved over time. These conditions are a determining factor in the final pattern formation, or lack thereof. [3]

4. Reaction Terms

4.1: **Turing's Reaction terms**. In Turing's original paper, *The Chemical Basis of Morphogenesis*, he did not include functional forms of reaction terms. Instead, he simply provided qualitative descriptions of the activator-inhibitor interactions and how they affect pattern formation. He knew that this was an important component of pattern formation, and those that came after him who researched the same topic came up with some functional reaction terms following Turing's descriptions. [12]

4.2: **Gierer-Meinhardt Model**. In 1972, Alfred Gierer and Hans Meinhardt created a model introducing a short-range activator and long-range inhibitor to show pattern formation from a homogenous initial system.

$$\partial u/\partial t = \nabla^2 u + a + u^2/v - bu$$
$$\partial v/\partial t = D_v\nabla^2 v + u2 - cv$$

Where a, b, c > 0 and D > 1. Notice that the first term in both of these equations resemble the first term in Turing's PDEs (given that $D_u = 1$). Everything that comes after the first term is part of the reaction term that Gierer and Meinhardt developed. Changing the parameters in these equations influences the type of pattern that is formed, but this system generally favors the formation of spots. By changing the initial conditions as shown below, one can observe stripe patterns and their instability.

$$u(0, x, y) = 1 + \cos(n\pi x/L), \quad v(0, x, y) = 1$$

A common way to get a stripey or labyrinth pattern using this model is to add a saturation term K to the self-activation term u^2/v.

$$\partial u/\partial t = \nabla^2 u + a + [u^2/v(1+Ku^2)] - bu$$

$$\partial v/\partial t = D\nabla^2 v + u^2 - cv$$

Altering the parameters and the value of K > 0 determines the kind of pattern that forms. [5]

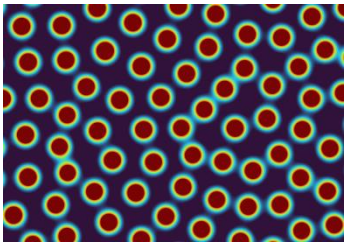

Figure 7: K = 0

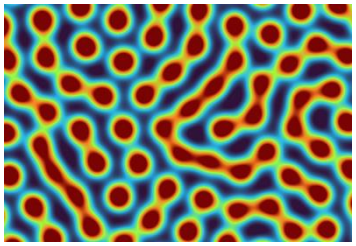

Figure 11: K = 0.002

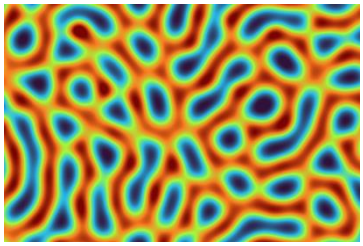

Figure 12: K = 0.003

4.3: **Brusselator Model**. The Brusselator Model uses two partial differential equations representing a two species chemical reaction (activator and inhibitor).

$$\begin{aligned} p_t &= D_p \Delta p + p^2 q + C - (K+1)p \\ q_t &= D_q \Delta q - p^2 q + Kp \end{aligned}$$
$$p(x, y, 0) = C + 0.1 \text{ for } 0 \leq x, y \leq 40$$
$$q(x, y, 0) = \frac{K}{C} + 0.2$$
$$u_n(x, y, t) = 0 \text{ on rectangle boundary, for all } t \geq 0$$

Where *p* represents the concentration of the activator and *q* represents the concentration of the inhibitor. The first two equations represent the PDEs, the second two represent the initial conditions, and the last one represents the boundary constraints. Pictured below are contour plots demonstrating the effect of changing the value of parameter K in the equations.

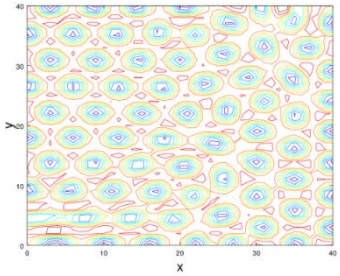

Figure 13: Brusselator contour plot at T=1000 and K=7

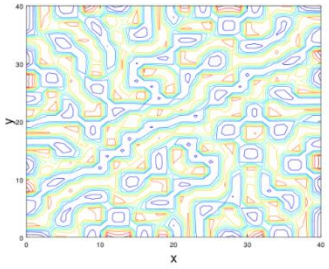

Figure 14: Brusselator contour plot at T=100 and K=9

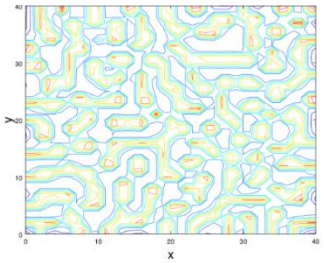

Figure 15: Brusselator contour plot at T=100 and K=11

Notice that the first plot appears more detailed than the other two. This is because the code used to create these plots was run until time = 1000 for the first plot and only time = 100 for the remaining two. The reasoning behind the discrepancy is that the code took too long to run for time = 1000, so the researchers cut down the time to 100 for the sake of saving time. Observe how when K = 7, the plot resembles spots. When K = 9, the plot resembles a labyrinth pattern with spots. Finally, when K = 11, the plot resembles a more defined labyrinth pattern. [7]

5. Advancements Since Turing

5.1: **Diffusiophoresis**. Researchers have found that Turing's explanation of diffusive transport results in patterns with shallower gradients than those that are found in nature. Recently, some researchers have proposed the idea that a process referred to as diffusiophoresis plays a role in pattern formation. Diffusiophoresis describes the movement of molecules in response to a concentration gradient of a separate chemical. In relation to Turing patterns, the molecules that move around in response to the system's chemical gradient are called chromatophores - defined as cells containing pigment. Diffusiophoresis causes the chromatophores to concentrate and clump together, creating color sharpening and more distinct patterns than seen in Turing's original model. [2]

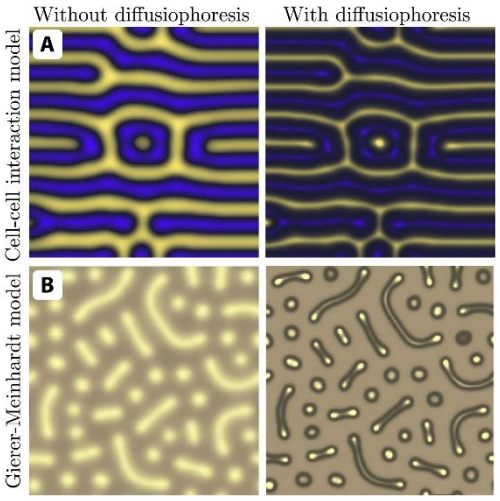

Figure 16: Comparing the patterns created by two different reaction-diffusion models with and without using diffusiophoresis

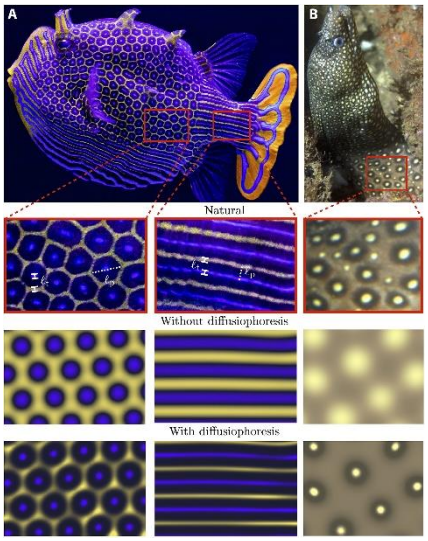

Figure 17: Comparing patterns found in nature vs. patterns simulated without diffusiophoresis vs. patterns simulated with diffusiophoresis

5.2: **Introducing Computers**. Since the introduction of computers as we know them today, researchers can investigate Turing patterns like never before. Computer can be used for:

1. Numerical simulation – computers can solve the reaction-diffusion equations that describe Turing patterns using numerical methods.
2. Parameter exploration – researchers can explore the effects of different parameter values on pattern formation by changing values such as the diffusion coefficients, reaction rates, initial conditions, stability.
3. Visualization – computers can generate visualizations using color maps, contour plots, surface plots, etc. to show the concentration of activator and inhibitor substances and the patterns they create.

4. Comparison with experimental data – compare simulated Turing patterns with those found within natural systems to gain insights into the mechanisms underlying pattern formation in the real world.
5. Parameter estimation and model fitting – iterative process that compares simulated patterns with experimental observations by adjusting the parameters to minimize any disparity between the two [6].

5.3: **Generative Art**. Generative art is an art discipline where either part of the piece or the entire piece is created with the help of an autonomous system. This autonomous system is non-human and can make decisions independently without any human input, like a computer system or algorithm. Jonathan McCabe is an Australian generative artist. In some of his pieces, he began to observe the characteristic spotted and striped patterns seen in Turing patterns. McCabe was already familiar with Turing patterns and decided to alter his program to specifically mimic a chemical system and observe what patterns arose. To do this, he devised a program that used pixels in place of cells. The program assigned random numbers to the pixels, each of which produced a color. Additionally, the number of pixels changed based on the ones around them, mimicking the characteristics of the activator and inhibitor in a system. Originally, McCabe saw basic Turing patterns like spots and stripes. Then, he began layering multiple Turing patterns on top of each other, creating multi-scale Turing patterns. He began to see Turing patterns mixed together or on top of each other, like large stripes comprised of small spots. Depending on what McCabe liked and disliked about the patterns that arose, he was able to tweak the algorithm or combine different algorithms into one. Many images resemble natural patterns such as iridescent fish scales, animal hide, blood vessels, and stained tissue samples. The beauty of generative art is that it is completely open to interpretation and every piece turns out differently but equally as beautiful as the last. [11]

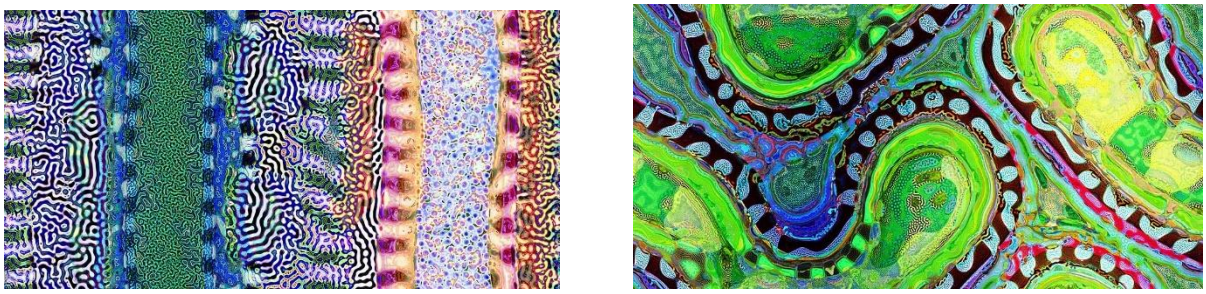

Figure 18: Two of McCabe's generative art pieces

6. Presentation Reflection

6.1: **Personal Reflection**. I was very happy with how this presentation turned out. After doing my last presentation on math in nature, I wanted to stick with something that I found interesting and hoped that others would too. It ended up taking me quite a long time to land on this topic. Originally, I was not going to do another presentation related to math and nature because I

thought that I had pretty much covered most of the information in my first presentation. So, I first started researching what other topics I could present on. However, nothing really piqued my interest like I wanted it to. Eventually, I landed on Turing patterns. I remember seeing something when I was doing research for my first presentation about how math can be related to animal patterns, but I never investigated it. When I went back and looked into it for my second presentation, I decided that it was an intriguing topic and something that I would like to share with the class. My biggest challenge when researching Turing patterns was to figure out a way to explain the biological topics. Obviously, this is a math class and no one is expected to know anything about biology. Personally, I have not taken biology in a while and there were some pretty complex biological mechanisms in pattern formation. However, I felt like I found some good papers that simplified the process and I think I was able to convey the information in an understandable way. The other difficulty I had was with the math behind Turing patterns. The math that I presented was an extremely simplified version of all the math I read about. I wanted to talk about the basics just enough to show the importance of math in describing pattern formation without overwhelming the audience. Even I felt overwhelmed when reading about it and I could barely understand it myself. However, I feel like I did a good job explaining the basics and emphasizing that there is so much more complexity behind pattern formation than I mentioned. There are still very important mathematical concepts that I simply did not go into detail about for the purposes of my presentation.

6.2: **Peer Feedback**. Looking over the Blackboard discussion posts, it seems like my peers found the topic as interesting as I did. Many people, including myself, did not realize that Turing dabbled in theoretical biology. I think the fact that Turing made these discoveries made the topic feel almost more significant to the class. Most people knew who Turing was or had at least heard of him, so it seemed to me that when they heard that this biological theory was proposed by such a familiar name in math, they appeared more interested than they might have been if it was someone they had never heard of. On the discussion board, some people asked me some questions that I was not sure how to answer. Interestingly, some of the questions were ones that I had asked myself, but never had the time to look into. When I did eventually get the chance to investigate, I found it difficult to get answers to certain inquisitions. A lot of the mechanisms behind Turing patterns are still being researched and I think there simply are not answers to all these questions yet. The fact that I was able to read my peers' questions and use the knowledge that I now have to venture a guess into why things are the way they are made me excited, though. My guesses are probably not completely accurate, but having enough knowledge about the topic to come up with my own ideas shows me that I really did take a lot away from this presentation and this class.

6.3: **Further Research**. The interdisciplinary nature of this field makes it difficult to pinpoint the idea to only biological systems. There is evidence that the mechanisms behind Turing patterns can be applied elsewhere within biology and outside of it. Research suggests that it could have applications in fields such as tissue engineering, biomimetic design, and self-organizing systems. It could also be related to processes including embryonic development and ecological dynamics. There is still a long way to go in the research of Turing patterns and its applications.

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